The representation theory of the wreath product of a finite group with the monoid of all partial functions on a finite set as an EI-category algebra

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Abstract

Let G be a finite group. We provide a description of the ordinary quiver of the complex monoid algebra of the wreath product $G \wr \mathrm{PT}_n$, where PT_n denotes the monoid of all partial functions on an n-element set. This description depends on the multiplicities of simple G-modules appearing in the decomposition of tensor products of simple G-modules. We also prove that the global dimension of this algebra is $n-1$. Both results are obtained by analyzing the associated Ehresmann EI-category related to the monoid. Finally, we describe the quiver of the algebra of the wreath product of G with the submonoid of all order-preserving partial functions.

1 Introduction

A central aim in the study of monoid representations is to connect them with the general representation theory of associative algebras. Investigating the invariants of the monoid algebra of a finite monoid M is of particular interest. Throughout this paper, all modules are over the field of complex numbers $\mathbb{C}$, so we consider the complex monoid algebra $\mathbb{C}M$.

Among invariants of an algebra, the (ordinary) quiver plays a foundational role. A standard way to present a finite-dimensional algebra is via a quiver, which is a directed graph, bound by a set of relations on its paths. In this context, the quiver can be viewed as the generators part in a generators-and-relations presentation of the algebra. Even without explicit knowledge of the relations, the quiver alone encodes a significant amount of data about the algebra.

Determining the quiver of a monoid algebra is a fundamental problem in the representation theory of finite monoids [31, Chapter 17]. To date, descriptions of the quiver have been obtained for many monoids and families of monoids [4, 13, 14, 15, 16, 17, 19, 22, 24, 26, 30, 32]. In particular, in [26], the author described the quiver of the monoid algebra $\mathbb{C}\,\mathrm{PT}_n$, where PT_n is the monoid of all partial functions on an n-element set. In this paper, we aim to generalize this result to the complex monoid algebra of the partial wreath product $G \wr \mathrm{PT}_n$ of any finite group G with PT_n.

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Partial wreath products of groups with (partial) transformation semigroups play a crucial role in the Krohn-Rhodes decomposition theory of finite automata and Krohn-Rhodes complexity theory [18]. This monoid was also studied recently in [2, Section 9].

To compute the quiver of the monoid algebra of $G \wr \mathrm{PT}_n$, we follow the general framework established in [26] for PT_n. However, the introduction of the finite group G into the structure introduces complications in the underlying representation theory.

After reviewing the necessary preliminaries in Section 2, we introduce several branching rules in Section 3 that will be used in our proof.

The computation of the quiver is carried out in Section 4. Since $\mathbb{C}(G \wr \mathrm{PT}_n)$ is a finite E-Ehresmann and right restriction semigroup, its algebra is isomorphic to that of its associated Ehresmann category.

This category is the wreath product of G with the category of all onto functions between subsets of an n-element set (see also [23]). This is an EI-category, meaning that every endomorphism monoid is a group. To determine the quiver of an EI-category algebra, a well-known method (see [16, Section 6.3.1] and [11]) reduces the problem to the representation theory of its endomorphism groups. We follow this approach here.

The endomorphism groups in this case are of the form $G \wr S_k$, where S_k is the symmetric group. This computation utilizes known branching rules for $G \wr S_k$ to establish our main result. The description of the quiver itself is given in Theorem 4.10 and relies on the decomposition of tensor products of simple G-modules.

We remark that the similar problem of finding the quiver of the wreath product of G with the category of injective functions between subsets of an n-element set was solved in [27, Section 6]. However, in that case, the structure of the group G plays a minimal role in the description.

In Section 5, we prove that the global dimension of $\mathbb{C}(G \wr \mathrm{PT}_n)$ is $n-1$ by applying the EI-category framework to lift known results on the global dimension of PT_n from [29].

Finally, in Section 6, we compute the quiver of the complex monoid algebra $\mathbb{C}(G \wr \mathrm{PO}_n)$, where PO_n is the monoid of all order-preserving partial functions on an n-element set. The method is identical to that applied in the case of PT_n, but the branching rules are simpler in this setting.

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2 Preliminaries

2.1 Partial wreath product

Let $\mathcal{A}$ be a small category. We denote by $\mathcal{A}^0$ and $\mathcal{A}^1$ the sets of objects and morphisms of $\mathcal{A}$, respectively. For $a, b \in \mathcal{A}^0$, we write $\mathcal{A}(a, b)$ for the hom-set of morphisms with domain a and range b. Recall that a monoid can be viewed as a category with a single object. Following [25], we denote by **Pfn** the category whose objects are finite sets and whose morphisms are partial functions, and by **Set** the subcategory consisting of total functions as morphisms.

Let G be a finite group with identity element 1_G, and let $\mathrm{PT}(\mathcal{X}, G)$ denote the set of all partial functions from $\mathcal{X}$ to G. For every $f \in \mathrm{PT}(\mathcal{X}, G)$, we denote by $\mathrm{dom}(f)$ its domain. The set $\mathrm{PT}(\mathcal{X}, G)$ is a monoid under pointwise multiplication, where for any $f_1, f_2 \in \mathrm{PT}(\mathcal{X}, G)$, the product $f_1 \cdot f_2$ has domain $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$ and is defined by:

$$(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) \quad \text{for all } x \in \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2).$$

The identity element of this monoid is the constant function mapping every $x \in \mathcal{X}$ to 1_G.

Let $H : \mathcal{A} \to \mathbf{Pfn}$ be a functor. Define a new category $G \wr_H \mathcal{A}$ in the following way. The set of objects is the same as the set of objects of $\mathcal{A}$, that is, $(G \wr_H \mathcal{A})^0 = \mathcal{A}^0$. Given two objects $a, b \in \mathcal{A}^0$, the hom-set $(G \wr_H \mathcal{A})(a, b)$ is

$$\{(f, m) \mid f \in \mathrm{PT}(H(a), G),\ m \in \mathcal{A}(a, b), \text{ where } \mathrm{dom}(f) = \mathrm{dom}(H(m))\}.$$

Now, given two morphisms $(f, m) \in (G \wr_H \mathcal{A})(a, b)$ and $(f', m') \in (G \wr_H \mathcal{A})(b, c)$ the composition is

$$(f', m') \cdot (f, m) = ((f' \circ H(m)) \cdot f, m'm).$$

It is routine to verify that this composition is well defined and that $G \wr_H \mathcal{A}$ is indeed a category. If id_a is the identity morphism of $a \in \mathcal{A}^0$ and $\mathbf{1}_{H(a)} : H(a) \to G$ is the function defined by $\mathbf{1}_{H(a)}(x) = 1_G$ for every $x \in H(a)$, then the identity morphism of the object $a \in (G \wr_H \mathcal{A})^0$ is $(\mathbf{1}_{H(a)}, \mathrm{id}_a)$. The category $G \wr_H \mathcal{A}$ is called the *partial wreath product* of G and $\mathcal{A}$ with respect to H. We will apply this construction in two special cases. In the first case, H is a functor $H : \mathcal{A} \to \mathbf{Set}$. In this case, the construction reduces to the standard wreath product of a group with a category (see, for example, [23, 34]). In the second case, $\mathcal{A}$ is a monoid M with identity element 1_M. Here the construction is well-known, even when G and M are semigroups. It appears in [5] in the language of transformation semigroups, and in [10] under the name of 0-wreath products. See also [2, Section 9.1] and references therein. In this case, a functor $H : M \to \mathbf{Pfn}$ is an *incomplete M-action.* That is, it consists of a set $\mathcal{X}$ and a monoid homomorphism $\varphi : M \to \mathrm{PT}_{\mathcal{X}}$. For $m \in M$ and $x \in \mathcal{X}$, we will write $m \bullet x$ instead of $\varphi(m)(x)$. For every $m \in M$, we write

$$\mathrm{dom}(m) = \mathrm{dom}(\varphi(m)) \subseteq \mathcal{X}.$$

The monoid M acts on the right of $\mathrm{PT}(\mathcal{X}, G)$ by $f * m = f \circ \varphi(m)$. Explicitly, for $x \in \mathcal{X}$, the partial function $f * m$ is given by

$$(f * m)(x) = \begin{cases} f(m \bullet x) & m \bullet x \text{ and } f(m \bullet x) \text{ are both defined} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In this case, the partial wreath product $G \wr_{\mathcal{X}} M$ is a monoid whose underlying set is

$$G \wr_{\mathcal{X}} M = \{(f, m) \mid m \in M, \quad f \in \mathrm{PT}(\mathcal{X}, G), \quad \mathrm{dom}(f) = \mathrm{dom}(m)\}.$$

The operation is given by

$$(f_1, m_1) \cdot (f_2, m_2) = ((f_1 * m_2) \cdot f_2, m_1 m_2),$$

where $(f_1 * m_2)(x) = f_1(m_2 \bullet x)$ and the identity element is $(\mathbf{1}_{\mathcal{X}}, 1_M)$.

Remark 2.1. Many authors adopt the convention of composing functions from left to right. Under this convention, the monoid M acts on the right of $\mathcal{X}$ and on the left of $\mathrm{PT}(\mathcal{X}, G)$. Consequently, the multiplication in $G \wr_{\mathcal{X}} M$ is often written in a different form in the literature.

The case where M is a group is of great importance. If $f : \mathcal{X} \to G$ is a (total) function, we denote by $f^{-1} : \mathcal{X} \to G$ the function defined by $f^{-1}(x) = (f(x))^{-1}$ for every $x \in \mathcal{X}$. If M is a group, then $G \wr_{\mathcal{X}} M$ is a group, and the inverse of $(f, m) \in G \wr_{\mathcal{X}} M$ is given by

$$(f, m)^{-1} = ((f * m^{-1})^{-1}, m^{-1}).$$

There is a natural incomplete action of the monoid $\mathrm{PT}_{\mathcal{X}}$ on the set $\mathcal{X}$. Formally, the action is given by the identity function $\varphi : \mathrm{PT}_{\mathcal{X}} \to \mathrm{PT}_{\mathcal{X}}$. In this case, we simply write $G \wr \mathrm{PT}_{\mathcal{X}}$ for $G \wr_{\mathcal{X}} \mathrm{PT}_{\mathcal{X}}$. If $\mathcal{X} = [n] = \{1, \ldots, n\}$, we denote the corresponding wreath product by $G \wr \mathrm{PT}_n$. In this special case, the wreath product has a natural description using matrices over $G \cup \{0\}$. For $\alpha \in \mathrm{PT}_n$ and $f \in \mathrm{PT}([n], G)$ with $\mathrm{dom}(f) = \mathrm{dom}(\alpha)$, we denote by $[f, \alpha]$ an $n \times n$ matrix over $G \cup \{0\}$ defined by

$$[f, \alpha]_{i,j} = \begin{cases} f(j) & \alpha(j) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{M}$ denote the set of all such matrices. Note that each $[f, \alpha] \in \mathcal{M}$ has at most one non-zero entry in each column, so matrix multiplication is well-defined. It is straightforward to verify that the function $\psi : G \wr \mathrm{PT}_n \to \mathcal{M}$ defined by $\psi((f, \alpha)) = [f, \alpha]$ is a monoid isomorphism.

2.2 Ehresmann semigroups

Let S be a semigroup and let $E \subseteq S$ be a subset of idempotents. We define two equivalence relations $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$ on S.

$$a \widetilde{\mathcal{L}}_E b \iff (\forall e \in E \quad be = b \Leftrightarrow ae = a)$$

$$a \widetilde{\mathcal{R}}_E b \iff (\forall e \in E \quad eb = b \Leftrightarrow ea = a).$$

A subset $E \subseteq S$ of idempotents is called a *subsemilattice* if it is a commutative subsemigroup. It is well known that any commutative semigroup of idempotents has the structure of a semilattice (i.e. a poset where every two elements have a meet) if one defines $a \leq b$ whenever $ab = ba = a$. A semigroup S with a subsemilattice $E \subseteq S$ is called *right E-Ehresmann* if every $\widetilde{\mathcal{L}}_E$-class contains a unique idempotent from E and $\widetilde{\mathcal{L}}_E$ is a right congruence. We denote the unique idempotent in the $\widetilde{\mathcal{L}}_E$-class of a by a^*. Note that a^* is the unique minimal element $e \in E$ such that $ae = a$. It is well known that $\widetilde{\mathcal{L}}_E$ is a right congruence if and only if the identity $(ab)^* = (a^*b)^*$ holds for every $a, b \in S$.

Dually, we can consider semigroups for which every $\widetilde{\mathcal{R}}_E$ class contains a unique idempotent. We denote the unique idempotent in the $\widetilde{\mathcal{R}}_E$ class of a by a^+. Such a semigroup is called left E-Ehresmann if $\widetilde{\mathcal{R}}_E$ is a left congruence, or equivalently if $(ab)^+ = (ab^+)^+$ for every $a, b \in S$. A semigroup S with a subsemilattice $E \subseteq S$ is called *E-Ehresmann* if it is both left and right E-Ehresmann. The semilattice E is also called the set of *projections* of S.

A right (left) Ehresmann semigroup S is called *right (respectively, left) restriction* if the "right ample" (respectively, "left ample") identity $ea = a(ea)^*$ (respectively, $ae = (ae)^+a$) holds for every $a \in S$ and $e \in E$.

For every Ehresmann semigroup S, we can associate a category $\mathbf{C}(S)$ in the following way. The object set of $\mathbf{C}(S)$ is the set E of projections and the morphisms of $\mathbf{C}(S)$ are in one-to-one correspondence with elements of S. For every $a \in S$, we associate a morphism $C(a) \in \mathbf{C}(S)^1$ with domain a^* and range a^+ and if the range of $C(a)$ is the domain of $C(b)$, the composition $C(b) \cdot C(a)$ is defined to be $C(ba)$. For more facts and proofs on Ehresmann semigroups and Ehresmann categories, the reader is referred to [6, 7].

2.3 Algebras and modules

Let A be an algebra. We will only discuss unital, finite-dimensional $\mathbb{C}$-algebras in this paper. Likewise, when we say that V is a module over A (or an A-module or an A-representation), we mean that V is a finite-dimensional left module over A. For $a \in A$ and $v \in V$, we write $a \bullet v$ for the action of a on the module element v. An A-module V is *simple* (or *irreducible*) if 0 is its only proper submodule. The *ordinary quiver* Q of a finite dimensional algebra A is a directed graph defined in the following way: The vertices of Q are in a one-to-one correspondence with the simple modules of A (up to isomorphism). If V_i and V_j are simple modules of A (identified with two vertices of the quiver), then the number of arrows from V_i to V_j is

$$\dim \operatorname{Ext}^1(V_i, V_j)$$

where $\operatorname{Ext}^1(V, -)$ is the first right derived functor of $\operatorname{Hom}(V, -)$, see [20, Chapters 6-7]. More about modules of algebras and quivers can be found in [1].

In this paper, we discuss complex algebras of finite categories. Let $\mathcal{A}$ be a finite category. The *category algebra* $\mathbb{C}\mathcal{A}$ is a vector space over $\mathbb{C}$ with the morphisms of $\mathcal{A}$ as its basis. It consists of all formal linear combinations:

$$\left\{ \sum_{i=1}^{n} k_i m_i \mid k_i \in \mathbb{C},\, m_i \in \mathcal{A}^1 \right\}.$$

The multiplication in $\mathbb{C}\mathcal{A}$ is the linear extension of the following:

$$m' \cdot m = \begin{cases} m'm & \text{if } m'm \text{ is defined in } \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

The algebra $\mathbb{C}\mathcal{A}$ has a unit element given by the sum of the identity morphisms of all objects in $\mathcal{A}$:

$$1_{\mathbb{C}\mathcal{A}} = \sum_{c \in \mathcal{A}^0} \mathrm{id}_c\,.$$

Since a monoid is a category with a single object, this construction naturally specializes to the definition of a *monoid algebra*. In this case, the algebra $\mathbb{C}M$ consists of all formal linear combinations of elements of the monoid M, with multiplication defined as the linear extension of the monoid operation.

2.4 Complex group representations

Let G be a finite group. If V is a $\mathbb{C}$G-module, we will usually simply say that V is a G-module (or a G-representation). Equivalently, a G-module is a pair (V, ρ) of a $\mathbb{C}$-vector space V and a group homomorphism $\rho : G \to \mathrm{GL}(V)$. We denote the set of simple modules of G (up to isomorphism) by $\operatorname{IRep} G$. It is well known that every G-module is a finite direct sum of simple modules and that the number of different simple G-modules (up to isomorphism) is the number of conjugacy classes of G. We denote the trivial module of any group G by tr_G. Recall that if V is a G-module, then $V^* = \operatorname{Hom}(V, \mathbb{C})$ is also a G-module with operation $(g \bullet \varphi)(v) = \varphi(g^{-1} \bullet v)$. Let U and V be G-modules. The inner tensor product $U \otimes V$ is again a G-module with action defined by $g \bullet (u \otimes v) = (g \bullet u) \otimes (g \bullet v)$ and extending linearly. Now, assume that U_1 and U_2 are modules of G_1 and G_2, respectively. The outer tensor product $U_1 \otimes U_2$ of U_1 and U_2 is the $(G_1 \times G_2)$-module where $(g_1, g_2) \bullet (u_1 \otimes u_2) = (g_1 \bullet u_1) \otimes (g_2 \bullet u_2)$. Although the two types of tensor product can be distinguished by context, we prefer to use a different notation for the outer tensor product, denoting it by $\boxtimes$. Similarly, the simple tensors of $U \boxtimes V$ will be denoted by $u \boxtimes v$. It is well known that $\operatorname{IRep}(G_1 \times G_2) = \{U \boxtimes V \mid U \in \operatorname{IRep} G_1, V \in \operatorname{IRep} G_2\}$. The *character* χ_U of the G-module (U, ρ) is the function $\chi_U : G \to \mathbb{C}$ defined by $\chi_U(g) = \operatorname{trace}(\rho(g))$. Recall that the multiplicity of $U \in \operatorname{IRep} G$ as a simple constituent in some G-module V is given by the inner product of characters

$$\langle \chi_U, \chi_V \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_U(g) \overline{\chi_V(g)}.$$

We may omit the subscript G when the group is clear from the context. Recall also that $\chi_{V^*}(g) = \overline{\chi_V(g)}$, $\chi_{U \boxtimes V}((g_1, g_2)) = \chi_U(g_1)\chi_V(g_2)$ and $\chi_{U \otimes V}(g) = \chi_U(g)\chi_V(g)$. In order to simplify notation, we will usually omit χ and write U also for the character of U. Hence the above inner product will be written as

$$\langle U, V \rangle = \frac{1}{|G|} \sum_{g \in G} U(g) \overline{V(g)}.$$

Let (U, ρ) be a G-module and let $H \leq G$ be a subgroup. The *restriction* of (U, ρ) to H is the H-module $(\operatorname{Res}^G_H U, \operatorname{Res}^G_H \rho)$ defined by

$$\operatorname{Res}^G_H \rho(h)(u) = \rho(h)(u)$$

that is, restricting the homomorphism to the subgroup H. Note that $\dim \operatorname{Res}^G_H U = \dim U$ and if U is a simple G-module, then $\operatorname{Res}^G_H U$ does not have to be a simple H-module. Let (U, ρ) be an H-module, the *induction* to G, denoted $(\operatorname{Ind}^G_H U, \operatorname{Ind}^G_H \rho)$ is the tensor product

$$\operatorname{Ind}^G_H U = \mathbb{C}G \underset{\mathbb{C}H}{\otimes} U$$

where the G action is given by

$$g \bullet (s \otimes u) = (gs) \otimes u$$

where $s \in \mathbb{C}G$ and $u \in U$. However, we will also use the following more concrete description. Choose $S = \{s_1, \ldots, s_l\}$ to be representatives of the left cosets of H in G (where $l = [G : H]$). Note that any element $g \in G$ can be written in a unique way as $g = s_i h$ where $s_i \in S$ and $h \in H$. Every element of $\operatorname{Ind}^G_H U$ is a formal sum of the form

$$\alpha_1(s_1, u_1) + \ldots + \alpha_l(s_l, u_l)$$

where $u_i \in U$ and $\alpha_i \in \mathbb{C}$. In other words, as a vector space $\operatorname{Ind}^G_H U$ is $\bigoplus_{i=1}^{l} U$, that is, l copies of U. The action is defined on elements of the form (s_i, u) by

$$g \bullet (s_i, u) = (s_j, h \bullet u)$$

where s_j and h are unique such that $gs_i = s_j h$. The required action is given by extending linearly. Note that $\dim \operatorname{Ind}^G_H U = [G : H] \dim U$. It is important to mention that the modules $\operatorname{Ind}^G_H U$ and $\operatorname{Res}^G_H V$ depend not only on the groups G and H but also on the specific embedding of H into G. Hence, we will have to give the specific embeddings when discussing these modules. Both induction and restriction are transitive and additive, that is, if $K \leq H \leq G$ then

$$\operatorname{Ind}^G_H \operatorname{Ind}^H_K U \cong \operatorname{Ind}^G_K U, \quad \operatorname{Ind}^G_H (U \oplus V) \cong \operatorname{Ind}^G_H U \oplus \operatorname{Ind}^G_H V$$

and

$$\operatorname{Res}_K^H \operatorname{Res}_H^G U \cong \operatorname{Res}_K^G U, \quad \operatorname{Res}_H^G(U \oplus V) \cong \operatorname{Res}_H^G U \oplus \operatorname{Res}_H^G V.$$

For the restriction this is a trivial statement, and for the induction the proof is [3, Propositions 1.1.10 and 1.1.11]. An important fact that relates induction to restriction is the following one (for a proof, see [3, Corollary 1.1.20]).

Theorem 2.2 (Frobenius reciprocity)**.** *Let $H \leq G$ and let U and V be G and H-modules respectively. Then the multiplicity of V in $\operatorname{Res}_H^G U$ equals the multiplicity of U in $\operatorname{Ind}_H^G V$.*

Using characters, Frobenius reciprocity can be written as the following equality

$$\langle \operatorname{Ind}_H^G V, U \rangle_G = \langle V, \operatorname{Res}_H^G U \rangle_H.$$

Let U be a G-module. Consider the *swap transformation* $S : U \otimes U \to U \otimes U$ defined on simple tensors by $S(u_1 \otimes u_2) = u_2 \otimes u_1$. We define the *symmetric square* $\operatorname{Sym}^2 U$ and the *alternating square* $\operatorname{Alt}^2 U$ as the following submodules of the tensor product $U \otimes U$:

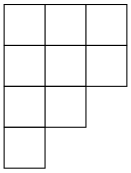
$$\begin{aligned} \operatorname{Sym}^2 U &= \{v \in U \otimes U \mid S(v) = v\}, \\ \operatorname{Alt}^2 U &= \{v \in U \otimes U \mid S(v) = -v\}. \end{aligned}$$

As G-modules, $U \otimes U \cong \operatorname{Sym}^2 U \oplus \operatorname{Alt}^2 U$. The characters of these modules are given by:

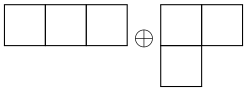
$$(\operatorname{Sym}^2 U)(g) = \frac{1}{2}\left(U(g)^2 + U(g^2)\right) \quad \text{and} \quad (\operatorname{Alt}^2 U)(g) = \frac{1}{2}\left(U(g)^2 - U(g^2)\right).$$

2.5 Representation theory of S_n and $G \wr S_n$

We will recall some elementary facts regarding the representation theory of the symmetric group. More details can be found in [9, 21]. Recall that an *integer composition* of n is a tuple $\lambda = [\lambda_1, \ldots, \lambda_k]$ of non-negative integers such that $\lambda_1 + \cdots + \lambda_k = n$ while an *integer partition* of n (denoted $\lambda \vdash n$) is an integer composition such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. From now on, when dealing with a partition λ we will write its elements in superscript $\lambda = [\lambda^1, \ldots, \lambda^k]$ because we want to reserve the subscript for multipartitions. Note that 0 has one partition, namely the empty partition, denoted by $\emptyset$. We can associate to any partition λ a graphical description called a *Young diagram*, which is a table with λ^i boxes in its i-th row. For instance, the Young diagram associated to the partition $[3, 3, 2, 1]$ of 9 is:



We will identify the two notions and regard integer partition and Young diagram as synonyms. It is well-known that simple modules of S_n are indexed by integer partitions of n. We denote the simple module associated to the partition λ (also called its *Specht module*) by S^λ. An explicit description of S^λ can be found in [21, Section 2.3]. It will be often convenient to draw the diagram λ instead of writing S^λ. For instance we may write



instead of: $S^{\lambda} \oplus S^{\delta}$ for partitions $\lambda = [3]$ and $\delta = [2, 1]$.

Let $\mathbf{n} = [n_1, \ldots, n_l]$ be an integer composition of n. A tuple $\Lambda = (\lambda_1, \ldots, \lambda_l)$ such that $\lambda_i \vdash n_i$ for every i is called a *multipartition* of n with l components. We also call it a multipartition of the composition $\mathbf{n}$ and denote this by $\Lambda \Vdash \mathbf{n}$. We define a *multi-Young diagram* to be a tuple of Young diagrams. As we identify partitions with Young diagrams, we also identify multipartitions with multi-Young diagrams.

Let G be a finite group with l conjugacy classes. It is well-known ([3, Theorem 2.6.1]) that multi-Young diagrams with n boxes and l components index the simple modules of the wreath product $G \wr S_n$. If $\Lambda \Vdash \mathbf{n}$ is a multi-Young diagram, then we denote by S^{Λ} its associated $G \wr S_n$-module.

3 Branching rules

Let G be a finite group with l conjugacy classes. Fix $\operatorname{IRep} G = \{U_1, \ldots U_l\}$ to be its set of simple modules. In this section we will describe several known branching rules for the group $G \wr S_n$ that we will need later on.

3.1 Restriction of $G \wr S_2$

Let $S_2 = \{\mathrm{id}, s\}$ be the symmetric group of order 2 where s will always be the swap permutation. We denote by tr_2 the trivial module of S_2 and by sgn_2 the sign module of S_2. The group $G \times S_2$ can be embedded in $G \wr S_2$ by $\varphi(g, \sigma) = ((g, g), \sigma)$. In this section, we study the restriction $\operatorname{Res}^{G \wr S_2}_{G \times S_2}$. Every simple module of $G \times S_2$ is of the form $U_r \boxtimes \mathrm{tr}_2$ or $U_r \boxtimes \mathrm{sgn}_2$ where $U_r \in \operatorname{IRep} G$ ($r \in \{1, \ldots l\}$). The description of simple modules of $G \wr S_2$ is more complicated, but well-known (see [3, Chapter 2] for the description of simple modules of $G \wr S_n$ in general). If $U_i, U_j \in \operatorname{IRep} G$ are two non-isomorphic simple modules of G then $U_i \boxtimes U_j$ is a $G \times G$ simple module. Clearly, $G \times G$ embeds in $G \wr S_2$ by $\varphi(g_1, g_2) = ((g_1, g_2), \mathrm{id})$. One type of simple $G \wr S_2$ module is obtained by induction $W_{i,j} = \operatorname{Ind}^{G \wr S_2}_{G \times G}(U_i \boxtimes U_j)$. This gives us $\binom{l}{2}$ simple modules. Note that $\dim W_{i,j} = 2 \dim U_i \dim U_j$. The module $W_{i,j}$ corresponds to the multipartition Λ with $\lambda_i = \lambda_j = [1]$ and $\lambda_k = \emptyset$ for $k \neq i, j$. Given $U_i \in \operatorname{IRep} G$, another type of simple module is the tensor product $U_i \boxtimes U_i$ where the action is

$$((g_1, g_2), \sigma) \bullet (u_1 \boxtimes u_2) = (g_{\sigma(1)} \bullet u_{\sigma(1)}) \boxtimes (g_{\sigma(2)} \bullet u_{\sigma(2)}).$$

We denote this simple module by W_i^+. This gives another l simple modules. The module W_i^+ corresponds to the multipartition Λ with $\lambda_i = [2]$ and $\lambda_k = \emptyset$ for $k \neq i$. Finally, we denote by $\operatorname{Inf}(\mathrm{sgn}_2)$ the *inflation* of sgn_2 to a $G \wr S_2$ module. This means that $((g_1, g_2), \sigma)$ acts like σ. The last l simple modules of $G \wr S_2$ are obtained by the tensor product $W_i^+ \otimes \operatorname{Inf}(\mathrm{sgn}_2)$ and we denote them by W_i^-. The module W_i^- corresponds to the multipartition Λ with $\lambda_i = [1, 1]$ and $\lambda_k = \emptyset$ for $k \neq i$. In total, we have $\frac{l^2+3l}{2}$ simple modules for $G \wr S_2$.

Remark 3.1. In the literature, the module W_i^+ is defined by the action

$$((g_1, g_2), \sigma) \bullet (u_1 \boxtimes u_2) = (g_1 \bullet u_{\sigma(1)}) \boxtimes (g_2 \bullet u_{\sigma(2)})$$

(in fact, with σ^{-1}, but $\sigma^{-1} = \sigma$ in our case.) This variation arises because, in the context of groups, the composition in the wreath product $G \wr M$ is defined by

$$(f_2, m_2) \cdot (f_1, m_1) = (f_2 \cdot (m_2 \star f_1), m_2 m_1)$$

where $(m \star f_1)(x) = f_1(m^{-1}x)$. As shown in [27, Lemma 6.3], the two definitions are isomorphic under the map

$$T(f, m) = (m \star f, m)$$

but it changes the concrete description of the module W_i^+. In any case, we are interested in the action of elements of the form $((g, g), \sigma)$. For such elements, the two actions coincide, ensuring there is no ambiguity in our context.

The characters of these modules are well-known and essential for computing the desired restriction. Recall that we use the same notation for a module and its character. If we take $((1_G, 1_G), \mathrm{id})$ and $((1_G, 1_G), s)$ as representatives of the cosets of $G \times G$ in $G \wr S_2$, it is easy to see that $((g, g), s)$ swaps the cosets and $((g, g), \mathrm{id})$ fixes them. The character of the induction is summation of the base character over the fixed cosets (see [21, Section 1.12]). Therefore, the character $W_{i,j}$ for an element $((g, g), \sigma) \in G \wr S_2$ is given by:

$$W_{i,j}((g, g), \sigma) = \begin{cases} 2U_i(g)U_j(g) & \text{if } \sigma = \mathrm{id}, \\ 0 & \text{if } \sigma = s. \end{cases}$$

For the modules W_i^+ and W_i^-, the character values on the elements $((g, g), \sigma)$ are given as follows (see [9, 4.3.10 (vi), p. 150]):

$$W_i^+((g, g), \sigma) = \begin{cases} U_i(g)^2 & \text{if } \sigma = \mathrm{id}, \\ U_i(g^2) & \text{if } \sigma = s, \end{cases}$$

and

$$W_i^-((g, g), \sigma) = \begin{cases} U_i(g)^2 & \text{if } \sigma = \mathrm{id}, \\ -U_i(g^2) & \text{if } \sigma = s. \end{cases}$$

Lemma 3.2. *The multiplicities of $U_k \boxtimes \mathrm{tr}_2$ and $U_k \boxtimes \mathrm{sgn}_2$ in $\mathrm{Res}^{G \wr S_2}_{G \times S_2} W_{i,j}$ are both equal to the multiplicity of U_k in $U_i \otimes U_j$.*

Proof. Let V be either the trivial module tr_2 or the sign module sgn_2 of S_2. The multiplicity of $U_k \boxtimes V$ in the restriction is:

$$\langle \mathrm{Res}^{G \wr S_2}_{G \times S_2} W_{i,j}, U_k \boxtimes V \rangle_{G \times S_2} = \frac{1}{2|G|} \sum_{g \in G} \sum_{\sigma \in S_2} W_{i,j}((g, g), \sigma) \overline{U_k(g) V(\sigma)}.$$

As $W_{i,j}((g, g), s) = 0$, the sum over $\sigma \in S_2$ only has a contribution from $\sigma = \mathrm{id}$. Since $V(\mathrm{id}) = 1$ for both the trivial and sign modules, the expression becomes:

$$\frac{1}{2|G|} \sum_{g \in G} W_{i,j}((g, g), \mathrm{id}) \overline{U_k(g)} = \frac{1}{2|G|} \sum_{g \in G} 2U_i(g)U_j(g)\overline{U_k(g)} = \langle U_i \otimes U_j, U_k \rangle_G,$$

which completes the proof. □

Lemma 3.3. *The multiplicities of* $U_k \boxtimes \mathrm{tr}_2$ *and* $U_k \boxtimes \mathrm{sgn}_2$ *in the restrictions* $\mathrm{Res}^{G\wr S_2}_{G\times S_2} W_i^+$ *and* $\mathrm{Res}^{G\wr S_2}_{G\times S_2} W_i^-$ *are given by the multiplicities of* U_k *in* $\mathrm{Sym}^2 U_i$ *and* $\mathrm{Alt}^2 U_i$ *as follows:*

1. $\langle \mathrm{Res}^{G\wr S_2}_{G\times S_2} W_i^+, U_k \boxtimes \mathrm{tr}_2\rangle_{G\times S_2} = \langle \mathrm{Sym}^2 U_i, U_k\rangle_G$
2. $\langle \mathrm{Res}^{G\wr S_2}_{G\times S_2} W_i^+, U_k \boxtimes \mathrm{sgn}_2\rangle_{G\times S_2} = \langle \mathrm{Alt}^2 U_i, U_k\rangle_G$
3. $\langle \mathrm{Res}^{G\wr S_2}_{G\times S_2} W_i^-, U_k \boxtimes \mathrm{tr}_2\rangle_{G\times S_2} = \langle \mathrm{Alt}^2 U_i, U_k\rangle_G$
4. $\langle \mathrm{Res}^{G\wr S_2}_{G\times S_2} W_i^-, U_k \boxtimes \mathrm{sgn}_2\rangle_{G\times S_2} = \langle \mathrm{Sym}^2 U_i, U_k\rangle_G$

Proof. We prove the first case; the others follow by similar character computations. By the definition of the inner product of characters on $G \times S_2$, we have:

$$\langle \mathrm{Res}^{G\wr S_2}_{G\times S_2} W_i^+, U_k \boxtimes \mathrm{tr}_2\rangle_{G\times S_2} = \frac{1}{2|G|}\sum_{g\in G}\left(W_i^+((g,g),\mathrm{id})\overline{U_k(g)} + W_i^+((g,g),s)\overline{U_k(g)}\right).$$

Substituting the character values $W_i^+((g,g),\mathrm{id}) = U_i(g)^2$ and $W_i^+((g,g),s) = U_i(g^2)$, we obtain:

$$\frac{1}{|G|}\sum_{g\in G}\left(\frac{U_i(g)^2 + U_i(g^2)}{2}\right)\overline{U_k(g)}.$$

The term in parentheses is precisely the character of $\mathrm{Sym}^2 U_i$. Thus, the expression reduces to $\langle \mathrm{Sym}^2 U_i, U_k\rangle_G$, which is the multiplicity of U_k in $\mathrm{Sym}^2 U_i$. □

3.2 Littlewood-Richardson rules for small additions

Let G be a group with l conjugacy classes. The group $(G \wr S_k) \times (G \wr S_r)$ is naturally embedded in $G \wr S_{k+r}$. If $f_1 : \{1,\dots,k\} \to G$ and $f_2 : \{1,\dots,r\} \to G$ we define $f : \{1,\dots,k+r\} \to G$ by

$$f(i) = \begin{cases} f_1(i) & i \le k \\ f_2(i-k) & i > k \end{cases}.$$

Then, the natural embedding $\varphi : G \wr S_k \times G \wr S_r \to G \wr S_{k+r}$ is defined by $\varphi((f_1,\sigma_1),(f_2,\sigma_2)) = (f, \sigma_1\sigma_2)$ where σ_1 (σ_2) can be regarded as an element of S_{k+r} that fixes $\{k+1,\dots,k+r\}$ ($\{1,\dots,k\}$). The branching rules for describing the induction from $G \wr S_k \times G \wr S_r$ to $G \wr S_{k+r}$ are known (see [8, Theorem 4.7] or [27, Theorem 4.5]), but we will give here only the cases for $r = 1, 2$ as this is what we need in this paper.

Let $\mathbf{k} = [k_1,\dots,k_l]$ be a composition of k and let $\Lambda \Vdash \mathbf{k}$ be a multi-Young diagram and let S^Λ be the associated $G \wr S_k$-module. Let U_i be a $G = G \wr S_1$-module. We can think of it as a multi-Young diagram with one box in the i-th component and all the other components are empty. Let $Y_i^+(\Lambda)$ be the set of multi-Young diagrams that can be obtained from Λ by adding one box at the i-th component. Conversely, let $Y_i^-(\Lambda)$ be the set of multi-Young diagrams obtained from Λ by removing one box from the i-th component.

Proposition 3.4. *The induction and restriction rules are as follows:*

1. ***Induction:***
$$\mathrm{Ind}_{(G\wr S_k)\times G}^{G\wr S_{k+1}}(S^\Lambda \boxtimes U_i) = \bigoplus_{\Gamma\in Y_i^+(\Lambda)} S^\Gamma$$

2. ***Restriction:*** *By Frobenius reciprocity, the restriction of a $G\wr S_{k+1}$-module S^Γ to the subgroup $(G\wr S_k)\times G$ is given by:*
$$\mathrm{Res}_{(G\wr S_k)\times G}^{G\wr S_{k+1}}(S^\Gamma) = \bigoplus_{i=1}^{l} \bigoplus_{\Lambda\in Y_i^-(\Gamma)} S^\Lambda \boxtimes U_i$$

For the case $r = 2$, we define the sets of multi-Young diagrams obtained by adding exactly two boxes to Λ:

- $Y_{i,j}^+(\Lambda)$: one box added to component i and one box added to component j $(i \neq j)$.
- $Y_{i,H^2}^+(\Lambda)$: a *horizontal strip* of two boxes added to component i. This means that we cannot add the two boxes in the same column.
- $Y_{i,V^2}^+(\Lambda)$: a *vertical strip* of two boxes added to component i. This means that we cannot add the two boxes in the same row.

Proposition 3.5. *The induction from $(G\wr S_k)\times(G\wr S_2)$ to $G\wr S_{k+2}$ for the various simple $G\wr S_2$-modules is given by:*
$$\mathrm{Ind}_{(G\wr S_k)\times(G\wr S_2)}^{G\wr S_{k+2}}(S^\Lambda \boxtimes W_{i,j}) = \bigoplus_{\Gamma\in Y_{i,j}^+(\Lambda)} S^\Gamma$$
$$\mathrm{Ind}_{(G\wr S_k)\times(G\wr S_2)}^{G\wr S_{k+2}}(S^\Lambda \boxtimes W_i^+) = \bigoplus_{\Gamma\in Y_{i,H^2}^+(\Lambda)} S^\Gamma$$
$$\mathrm{Ind}_{(G\wr S_k)\times(G\wr S_2)}^{G\wr S_{k+2}}(S^\Lambda \boxtimes W_i^-) = \bigoplus_{\Gamma\in Y_{i,V^2}^+(\Lambda)} S^\Gamma$$

4 The quiver of $\mathbb{C}(G\wr \mathrm{PT}_n)$

4.1 Ehresmann structure

For every $X \subseteq [n] = \{1,\dots n\}$ we define id_X to be the partial identity of the set X
$$\mathrm{id}_X(x) = \begin{cases} x & \text{if } x\in X,\\ \text{undefined} & \text{otherwise,}\end{cases}$$

and $\mathbf{1}_X : [n]\to G$ to be the function
$$\mathbf{1}_X(x) = \begin{cases} 1_G & \text{if } x\in X,\\ \text{undefined} & \text{otherwise.}\end{cases}$$

Set $\mathcal{E} = \{(\mathbf{1}_X, \mathrm{id}_X) \mid X \subseteq [n]\}$ and note that this is a subsemilattice of $G \wr \mathrm{PT}_n$. It is routine to verify that the idempotent $(\mathbf{1}_X, \mathrm{id}_X)$ is a left (right) identity of $(f, \alpha) \in G \wr \mathrm{PT}_n$ if and only if $\mathrm{im}(\alpha) \subseteq X$ (respectively, $\mathrm{dom}(\alpha) \subseteq X$). Therefore, we have that two elements (f, α) and (g, β) of $G \wr \mathrm{PT}_n$ are $\widetilde{\mathcal{L}}_{\mathcal{E}}$-related if and only if $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$, and they are $\widetilde{\mathcal{R}}_{\mathcal{E}}$-related if and only if $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$. In other words, if $X = \mathrm{dom}(\alpha)$ and $Y = \mathrm{im}(\alpha)$ then

$$(f, \alpha)^* = (\mathbf{1}_X, \mathrm{id}_X), \quad (f, \alpha)^+ = (\mathbf{1}_Y, \mathrm{id}_Y).$$

In [2, Proposition 9.11], it was proved that $G \wr \mathrm{PT}_n$ is a right $\mathcal{E}$-restriction monoid. It is also easy to prove that the left congruence condition holds.

Lemma 4.1. *The relation $\widetilde{\mathcal{R}}_{\mathcal{E}}$ is a left congruence on $G \wr \mathrm{PT}_n$.*

Proof. Let $(f_1, \alpha_1), (f_2, \alpha_2) \in G \wr \mathrm{PT}_n$ such that $(f_1, \alpha_1)\widetilde{\mathcal{R}}_{\mathcal{E}}(f_2, \alpha_2)$. This is equivalent to $\mathrm{im}(\alpha_1) = \mathrm{im}(\alpha_2)$. Let (f_3, α_3) be any element in $G \wr \mathrm{PT}_n$. Note that $\mathrm{im}(\alpha_3\alpha_1) = \mathrm{im}(\alpha_3\alpha_2)$ so

$$(f_3, \alpha_3)(f_1, \alpha_1) = ((f_3 * \alpha_1) \cdot f_1, \alpha_3\alpha_1)\tilde{\mathcal{R}}_{\mathcal{E}}((f_3 * \alpha_2) \cdot f_2, \alpha_3\alpha_2) = (f_3, \alpha_3)(f_2, \alpha_2).$$

Thus, $\widetilde{\mathcal{R}}_{\mathcal{E}}$ is a left congruence. □

It follows that $G \wr \mathrm{PT}_n$ is a $\mathcal{E}$-Ehresmann and right restriction monoid. This will be crucial in view of the following fact.

Theorem 4.2 ([28, Theorem 1.5])**.** *Let M be a finite E-Ehresmann and right restriction monoid and let $\mathbf{C}(M)$ be its associated Ehresmann category. Then, for every unital commutative ring $\Bbbk$ there is an isomorphism of algebras $\Bbbk M \simeq \Bbbk\mathbf{C}(M)$*

Therefore, we can switch to studying the representation theory of the associated category $\mathbf{C}(G \wr \mathrm{PT}_n)$. We start by describing it. The objects of $\mathbf{C}(G \wr \mathrm{PT}_n)$ are in one-to-one correspondence with the elements of $\mathcal{E}$. Thus, the objects are of the form $(\mathbf{1}_X, \mathrm{id}_X)$ for $X \subseteq [n]$. For two subsets $X, Y \subseteq [n]$, the hom-set $\mathbf{C}(G \wr \mathrm{PT}_n)((\mathbf{1}_X, \mathrm{id}_X), (\mathbf{1}_Y, \mathrm{id}_Y))$ is identified with the elements $(f, \alpha) \in G \wr \mathrm{PT}_n$ such that

$$(f, \alpha)^* = (\mathbf{1}_X, \mathrm{id}_X) \quad \text{and} \quad (f, \alpha)^+ = (\mathbf{1}_Y, \mathrm{id}_Y).$$

Note that in this case, $X = \mathrm{dom}(\alpha)$ and $Y = \mathrm{im}(\alpha)$. We denote by $C(f, \alpha)$ the morphism associated with (f, α).

Let E_n be the category defined as follows. The objects of E_n are subsets $X \subseteq [n]$. For $X, Y \subseteq [n]$, the hom-set $\mathrm{E}_n(X, Y)$ contains all the onto (total) functions $\alpha \colon X \to Y$. Let G be a group. We denote by $G \wr \mathrm{E}_n$ the wreath product $G \wr_H \mathrm{E}_n$, where $H \colon \mathrm{E}_n \to \mathbf{Set}$ is the inclusion functor.

Proposition 4.3. *There is an isomorphism of categories $\mathbf{C}(G \wr \mathrm{PT}_n) \simeq G \wr \mathrm{E}_n$.*

Proof. It follows immediately from the above discussion. Formally, an isomorphism

$$\psi \colon \mathbf{C}(G \wr \mathrm{PT}_n) \to G \wr \mathrm{E}_n$$

is defined by $\psi((\mathbf{1}_X, \mathrm{id}_X)) = X$ and $\psi(C(f, \alpha)) = (f, \alpha)$. Note that if $C(f, \alpha)$ is a morphism in $\mathbf{C}(G \wr \mathrm{PT}_n)((\mathbf{1}_X, \mathrm{id}_X), (\mathbf{1}_Y, \mathrm{id}_Y))$, then $\mathrm{dom}(\alpha) = X$ and $\mathrm{im}(\alpha) = Y$, so α is indeed a total onto function $\alpha \colon X \to Y$. Moreover, $\mathrm{dom}(f) = X$, so $f \in G^X$ as required in the definition of $G \wr \mathrm{E}_n$. It is easy to see now that ψ is an isomorphism. □

4.2 The skeleton

If $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories, then their algebras are Morita equivalent (see [33, Proposition 2.2]). Since the quiver of an algebra is an invariant of Morita equivalence, we can switch our attention to a simpler category which is equivalent to $G \wr \mathrm{E}_n$.

We can take a full subcategory with one object from every isomorphism class in $G \wr \mathrm{E}_n$. This category is called the *skeleton* of $G \wr \mathrm{E}_n$. To describe it, we first have to characterize which objects in $G \wr \mathrm{E}_n$ are isomorphic.

Lemma 4.4. *Two objects X, Y in $G \wr \mathrm{E}_n$ are isomorphic if and only if $|X| = |Y|$.*

Proof. First, note that if $|X| < |Y|$, then the hom-set $(G \wr \mathrm{E}_n)(X, Y)$ is empty because there are no onto functions from X to Y. If X and Y are isomorphic, then both $(G \wr \mathrm{E}_n)(X, Y)$ and $(G \wr \mathrm{E}_n)(Y, X)$ are non-empty, which implies $|X| = |Y|$.

Conversely, if $|X| = |Y|$, we can take any invertible function $\alpha \colon X \to Y$, and we claim that $(\mathbf{1}_X, \alpha)$ is an isomorphism with inverse $(\mathbf{1}_Y, \alpha^{-1})$. Indeed,

$$\begin{aligned}(\mathbf{1}_Y, \alpha^{-1}) \cdot (\mathbf{1}_X, \alpha) &= ((\mathbf{1}_Y * \alpha) \cdot \mathbf{1}_X, \alpha^{-1}\alpha) \\ &= ((\mathbf{1}_Y * \alpha) \cdot \mathbf{1}_X, \mathrm{id}_X).\end{aligned}$$

Note that since $\mathrm{im}(\alpha) = Y$, we have $\mathbf{1}_Y * \alpha = \mathbf{1}_X$. Therefore,

$$(\mathbf{1}_Y, \alpha^{-1}) \cdot (\mathbf{1}_X, \alpha) = (\mathbf{1}_X \cdot \mathbf{1}_X, \mathrm{id}_X) = (\mathbf{1}_X, \mathrm{id}_X),$$

which is the identity morphism of the object X. Likewise, since $\alpha\alpha^{-1} = \mathrm{id}_Y$, it follows that $(\mathbf{1}_X, \alpha) \cdot (\mathbf{1}_Y, \alpha^{-1}) = (\mathbf{1}_Y, \mathrm{id}_Y)$. □

Denote by $G \wr \mathrm{SE}_n$ the full subcategory of $G \wr \mathrm{E}_n$ whose objects are the sets $[0] = \emptyset$ and $[k] = \{1, \ldots, k\}$ for $1 \leq k \leq n$. As the notation suggests, this category can be identified with the wreath product of G with the category SE_n, where the set of objects is $\{[k] \mid 0 \leq k \leq n\}$ and the morphisms are total onto functions. Following the discussion above, to determine the quiver of the original algebra, it suffices to focus our attention on the quiver of the algebra $\mathbb{C}(G \wr \mathrm{SE}_n)$.

It is also convenient to describe $G \wr \mathrm{SE}_n$ using matrices. Let $\mathcal{D}_n$ be the category defined as follows. The set of objects of $\mathcal{D}_n$ is $\{[k] \mid 0 \leq k \leq n\}$. For $0 \leq k, r \leq n$, the hom-set $\mathcal{D}_n([k], [r])$ consists of all $r \times k$ matrices over $G \cup \{0\}$ with exactly one non-zero element in every column and at least one non-zero element in every row. Composition of morphisms is given by standard matrix multiplication, which is well-defined because each column contains only one group element.

It is straightforward to see that there is an isomorphism of categories $G \wr \mathrm{SE}_n \simeq \mathcal{D}_n$. To each morphism $(f, \alpha) \in (G \wr \mathrm{SE}_n)([k], [r])$, we associate an $r \times k$ matrix $[f, \alpha]$ defined by:

$$[f, \alpha]_{i,j} = \begin{cases} f(j) & \text{if } \alpha(j) = i, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, the j-th column of $[f, \alpha]$ contains a unique non-zero element in the $\alpha(j)$-th row. The condition that α is surjective implies that every row contains at least one non-zero element. It is routine to verify that this assignment respects composition and thus defines an isomorphism of categories.

4.3 The quiver of a skeletal EI-category algebra

A category is called an *EI-category* if its endomorphism monoids are groups. In other words, in an EI-category, every endomorphism is an isomorphism.

The category $G \wr \mathrm{SE}_n$ is an EI-category because for any morphism $(f, \alpha)\colon [k] \to [k]$, the map α is a surjective map from a finite set to itself, which is necessarily a bijection. Therefore, the endomorphism monoid of an object $[k]$ is the group $G \wr S_k$.

The problem of finding the ordinary quiver of the complex algebra of a skeletal EI-category can be reduced to a problem in group representation theory. We define a few concepts below and then state the relevant theorem.

For any finite set X, we denote by $\mathbb{C}X$ (or $\mathbb{C}[X]$) the complex vector space consisting of all formal linear combinations of elements of X. If a group G acts on X, then $\mathbb{C}X$ naturally becomes a $\mathbb{C}G$-module called a *permutation module*.

Let $\mathcal{A}$ be a finite EI-category. A morphism $m \in \mathcal{A}^1$ is called *irreducible* if it is not an isomorphism, and whenever $m = m_1 m_2$, either m_1 or m_2 is an isomorphism. We denote the set of irreducible morphisms from object c to object c' by $\mathrm{Irr}(\mathcal{A})(c, c')$. Recall that we denote the set of simple modules of a group G by $\mathrm{IRep}(G)$.

Theorem 4.5 ([16, Theorem 6.13], [11, Theorem 4.7])**.** *Let $\mathcal{A}$ be a finite skeletal EI-category and let Q be the quiver of $\mathbb{C}\mathcal{A}$. Then:*

1. *The set of vertices of Q is given by*

$$\bigsqcup_{c \in \mathcal{A}^0} \mathrm{IRep}(\mathcal{A}(c, c)).$$

2. *The vector space $\mathbb{C}[\mathrm{Irr}(\mathcal{A})(c, c')]$ can be viewed as an $(\mathcal{A}(c', c') \times \mathcal{A}(c, c))$-module with the action given by $(h, g) \bullet f = hfg^{-1}$. For $V \in \mathrm{IRep}(\mathcal{A}(c, c))$ and $U \in \mathrm{IRep}(\mathcal{A}(c', c'))$, the number of arrows from V to U is the multiplicity of $U \otimes V^*$ as a simple constituent in $\mathbb{C}[\mathrm{Irr}(\mathcal{A})(c, c')]$.*

In view of the theorem above, the vertices of the quiver Q of $\mathbb{C}(G \wr \mathrm{SE}_n)$ are indexed by the simple modules of the automorphism groups $G \wr S_k$ for each object $[k]$ in $G \wr \mathrm{SE}_n$. Since G has l conjugacy classes, these modules are indexed by multipartitions (or equivalently, multi-Young diagrams) Λ with k boxes and l components, where k varies from 0 to n.

4.4 Irreducible morphisms

The next step for using Theorem 4.5 is identifying the irreducible morphisms of $G \wr \mathrm{SE}_n$.

Lemma 4.6. *The irreducible morphisms of $G \wr \mathrm{SE}_n$ are precisely the morphisms from $[k+1]$ to $[k]$ for $0 \leq k < n$. In other words,*

$$\mathrm{Irr}(G \wr \mathrm{SE}_n)([p], [k]) = \begin{cases} G \wr \mathrm{SE}_n([p], [k]) & \text{if } p = k+1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. It is clear that every morphism (f, α) from $[k+1]$ to $[k]$ is irreducible. Indeed, if one decomposes $(f, \alpha) = (f_1, \alpha_1) \cdot (f_2, \alpha_2)$, the size of the sets implies that one of the factors must be an endomorphism and hence an isomorphism.

Now, if $(f,\alpha) \in G \wr \mathrm{SE}_n([p],[k])$ where $p > k+1$, then it is known that we can write $\alpha = \alpha_1\alpha_2$ for some onto functions $\alpha_2 \colon [p] \to [k+1]$ and $\alpha_1 \colon [k+1] \to [k]$ (see [26, Lemma 3.3]). In this case, $(\mathbf{1}_{[k+1]}, \alpha_1)$ and (f,α_2) are both well-defined morphisms which are not endomorphisms and therefore are not invertible. Finally,

$$(\mathbf{1}_{[k+1]}, \alpha_1) \cdot (f, \alpha_2) = (\mathbf{1}_{[k+1]} * \alpha_2 \cdot f, \alpha_1\alpha_2) = (\mathbf{1}_{[p]} \cdot f, \alpha_1\alpha_2) = (f,\alpha),$$

so (f,α) is not irreducible. $\square$

Let $V \in \mathrm{IRep}(G \wr S_p)$ and $U \in \mathrm{IRep}(G \wr S_k)$. If $p \neq k+1$, there are no arrows in the quiver of $\mathbb{C}(G \wr \mathrm{SE}_n)$ from V to U because $\mathrm{Irr}(G \wr \mathrm{SE}_n)([p],[k])$ is empty. Consequently, we focus on the case $p = k+1$ and examine the module $\mathbb{C}[\mathrm{Irr}(G \wr \mathrm{SE}_n)([k+1],[k])]$ under the action of $G \wr S_k \times G \wr S_{k+1}$ as described in Theorem 4.5.

For convenience, we denote the set of irreducible morphisms by

$$X = \mathrm{Irr}(G \wr \mathrm{SE}_n)([k+1],[k]) = \{(f,\alpha) \mid \alpha \colon [k+1] \to [k] \text{ is onto}, f \colon [k+1] \to G\}.$$

The group $G \wr S_k \times G \wr S_{k+1}$ acts on X via

$$((f_1,\sigma_1),(f_2,\sigma_2)) \bullet (f,\alpha) = (f_1,\sigma_1) \cdot (f,\alpha) \cdot (f_2,\sigma_2)^{-1},$$

where $f_1 \colon [k] \to G$, $f_2 \colon [k+1] \to G$, $\sigma_1 \in S_k$, and $\sigma_2 \in S_{k+1}$. The module of interest is the linearization $\mathbb{C}X$, which is a permutation module for this action.

4.5 Description of the action and stabilizer

Lemma 4.7. *The action of $G \wr S_k \times G \wr S_{k+1}$ on X is transitive.*

Proof. We show that every element of X lies in the orbit of $(\mathbf{1}_{[k+1]}, \mathrm{dec})$, where $\mathrm{dec} \colon [k+1] \to [k]$ is defined by

$$\mathrm{dec}(i) = \begin{cases} i & \text{if } i \leq k, \\ k & \text{if } i = k+1. \end{cases}$$

Let $(f,\alpha) \in X$. It is easy to verify (also mentioned in [26, Section 3]) that the action of $S_k \times S_{k+1}$ on the set of surjective maps from $[k+1]$ to $[k]$ is transitive. Thus, there exist $\sigma_1 \in S_k$ and $\sigma_2 \in S_{k+1}$ such that $\sigma_1 \,\mathrm{dec}\, \sigma_2 = \alpha$.

Taking $(\mathbf{1}_{[k]}, \sigma_1) \in G \wr S_k$ and $(f,\sigma_2)^{-1} \in G \wr S_{k+1}$, we observe that:

$$\begin{aligned}(\mathbf{1}_{[k]}, \sigma_1) \cdot (\mathbf{1}_{[k+1]}, \mathrm{dec}) \cdot \left((f,\sigma_2)^{-1}\right)^{-1} &= (\mathbf{1}_{[k]} * \mathrm{dec} \cdot \mathbf{1}_{[k+1]}, \sigma_1 \,\mathrm{dec}) \cdot (f,\sigma_2) \\ &= (\mathbf{1}_{[k+1]}, \sigma_1 \,\mathrm{dec}) \cdot (f,\sigma_2) \\ &= (\mathbf{1}_{[k+1]} * \sigma_2 \cdot f, \sigma_1 \,\mathrm{dec}\, \sigma_2) \\ &= (\mathbf{1}_{[k+1]} \cdot f, \sigma_1 \,\mathrm{dec}\, \sigma_2) \\ &= (f,\alpha).\end{aligned}$$

Thus, the action is indeed transitive. $\square$

Let K be the stabilizer of $(\mathbf{1}_{[k+1]}, \mathrm{dec})$ under the action of $G \wr S_k \times G \wr S_{k+1}$ described above, and let tr_K denote the trivial module of K. Since $\mathbb{C}X$ is a permutation module arising from a transitive group action, it follows that

$$\mathbb{C}X \simeq \mathrm{Ind}_K^{G \wr S_k \times G \wr S_{k+1}}(\mathrm{tr}_K).$$

To proceed, we must characterize the stabilizer K more explicitly.

For the action of $S_k \times S_{k+1}$ on the set of surjective maps from $[k+1]$ to $[k]$, it is known that the stabilizer of dec is the subgroup

$$\{(\sigma, \sigma\tau) \mid \sigma \in S_{k-1},\, \tau \in S_{\{k,k+1\}}\}$$

(see [26, Lemma 3.5]). For any function $f \in G^{[k]}$, let $\hat{f} \in G^{[k+1]}$ be the function defined by

$$\hat{f}(i) = \begin{cases} f(i) & \text{if } i \leq k, \\ f(k) & \text{if } i = k+1. \end{cases}$$

Note, in particular, that $\hat{f}(k+1) = \hat{f}(k)$.

Recall that we can view any element $\sigma \in S_{k-1}$ as an element of S_k that fixes k.

Lemma 4.8. *The stabilizer K of $(\mathbf{1}_{[k+1]}, \mathrm{dec})$ under the action of $G \wr S_k \times G \wr S_{k+1}$ is given explicitly by*

$$K = \{((f, \sigma), (\hat{f}, \sigma\tau)) \mid \sigma \in S_{k-1},\, \tau \in S_{\{k,k+1\}},\, f \in G^{[k]}\}.$$

Proof. An element $((f, \sigma), (h, \epsilon))$ is in K if and only if

$$(f, \sigma) \cdot (\mathbf{1}_{[k+1]}, \mathrm{dec}) \cdot (h, \epsilon)^{-1} = (\mathbf{1}_{[k+1]}, \mathrm{dec}),$$

or equivalently,

$$(f, \sigma) \cdot (\mathbf{1}_{[k+1]}, \mathrm{dec}) = (\mathbf{1}_{[k+1]}, \mathrm{dec}) \cdot (h, \epsilon).$$

Applying the product rule for the wreath product on both sides, we obtain

$$(f * \mathrm{dec} \cdot \mathbf{1}_{[k+1]}, \sigma\, \mathrm{dec}) = (\mathbf{1}_{[k+1]} * \epsilon \cdot h, \mathrm{dec}\, \epsilon).$$

Since $\mathbf{1}_{[k+1]}$ is the identity for the pointwise product, this simplifies to the condition

$$(f * \mathrm{dec}, \sigma\, \mathrm{dec}) = (h, \mathrm{dec}\, \epsilon).$$

The map equality $\sigma\, \mathrm{dec} = \mathrm{dec}\, \epsilon$ implies that (σ, ϵ) is in the stabilizer of dec under the action of $S_k \times S_{k+1}$. It follows that $\sigma \in S_{k-1}$ and $\epsilon = \sigma\tau$ for some $\tau \in S_{\{k,k+1\}}$.

Finally, the function equality $f * \mathrm{dec} = h$ is equivalent to $h = \hat{f}$. Indeed, for $i \leq k$, we have $h(i) = f(\mathrm{dec}(i)) = f(i)$, and for $i = k+1$, we have $h(k+1) = f(\mathrm{dec}(k+1)) = f(k)$. This completes the proof. ☐

For any $f \in G^{[k-1]}$ and $g \in G$, we define a function $f^g \in G^{[k]}$ by

$$f^g(i) = \begin{cases} f(i) & i \leq k-1 \\ g & i = k \end{cases}.$$

Note that every $h \in G^{[k]}$ can be uniquely written as $h = f^g$ for some $f \in G^{[k-1]}$ and $g \in G$.

In what follows, it will be useful to understand how elements of K correspond to matrices over $G \cup \{0\}$. Given an element (f^g, σ) with $\sigma \in S_{k-1}$, its associated matrix consists of an $(k-1)\times(k-1)$ block A associated with (f, σ) and a 1×1 block containing g. Thus, (f^g, σ) corresponds to a block-diagonal matrix:

$$A \oplus (g) = \begin{pmatrix} A & 0 \\ 0 & g \end{pmatrix}.$$

For the element $(\widehat{f^g}, \sigma\tau)$, the associated matrix similarly decomposes into two blocks of sizes $(k-1) \times (k-1)$ and 2×2. The first block is A, and the second is the 2×2 block gP_τ, where P_τ is the permutation matrix associated with τ:

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & gP_\tau \end{pmatrix}.$$

Lemma 4.9. *There is an isomorphism of groups*

$$K \simeq (G \wr S_{k-1}) \times G \times S_2.$$

Proof. Identify S_2 with $S_{\{k,k+1\}}$. We define a map $\psi : (G \wr S_{k-1}) \times G \times S_2 \to K$ by

$$\psi((f, \sigma), g, \tau) = \left((f^g, \sigma), (\widehat{f^g}, \sigma\tau)\right).$$

By Lemma 4.8, it is clear that ψ is a bijection. It remains to show that ψ is a group homomorphism. Indeed,

$$\begin{aligned}
\psi(((f_1, \sigma_1), g_1, \tau_1) \cdot ((f_2, \sigma_2), g_2, \tau_2)) &= \psi(((f_1, \sigma_1) \cdot (f_2, \sigma_2)), g_1 g_2, \tau_1 \tau_2) \\
&= \psi((f_1 * \sigma_2 \cdot f_2, \sigma_1 \sigma_2), g_1 g_2, \tau_1 \tau_2) \\
&= (((f_1 * \sigma_2 \cdot f_2)^{g_1 g_2}, \sigma_1 \sigma_2), (\widehat{(f_1 * \sigma_2 \cdot f_2)^{g_1 g_2}}, \sigma_1 \sigma_2 \tau_1 \tau_2))
\end{aligned}$$

On the other hand

$$\begin{aligned}
\psi((f_1, \sigma_1), g_1, \tau_1) \cdot \psi((f_2, \sigma_2), g_2, \tau_2) &= ((f_1^{g_1}, \sigma_1), (\widehat{f_1^{g_1}}, \sigma_1 \tau_1)) \cdot ((f_2^{g_2}, \sigma_2), (\widehat{f_2^{g_2}}, \sigma_2 \tau_2)) \\
&= ((f_1^{g_1} * \sigma_2 \cdot f_2^{g_2}, \sigma_1 \sigma_2), (\widehat{f_1^{g_1}} * \sigma_2 \tau_2 \cdot \widehat{f_2^{g_2}}, \sigma_1 \tau_1 \sigma_2 \tau_2)).
\end{aligned}$$

First note that

$$\sigma_1 \sigma_2 \tau_1 \tau_2 = \sigma_1 \tau_1 \sigma_2 \tau_2$$

because τ_1 and σ_2 have disjoint supports and thus commute as elements of S_{k+1}. Next, for $i \leq k-1$, we have

$$(f_1 * \sigma_2 \cdot f_2)^{g_1 g_2}(i) = f_1(\sigma_2(i)) \cdot f_2(i) = f_1^{g_1}(\sigma_2(i)) \cdot f_2^{g_2}(i)$$

because $\sigma_2(i) \leq k-1$ as well. For $i = k$, we have

$$(f_1 * \sigma_2 \cdot f_2)^{g_1 g_2}(k) = g_1 g_2 = f_1^{g_1}(k) \cdot f_2^{g_2}(k) = f_1^{g_1}(\sigma_2(k)) \cdot f_2^{g_2}(k)$$

since σ_2 fixes k. This establishes that

$$(f_1 * \sigma_2 \cdot f_2)^{g_1 g_2} = f_1^{g_1} * \sigma_2 \cdot f_2^{g_2}.$$

Proving the last equality is similar. For $i \leq k-1$, we have again

$$\widehat{(f_1 * \sigma_2 \cdot f_2)^{g_1 g_2}}(i) = f_1(\sigma_2(i)) \cdot f_2(i)$$

and

$$\begin{aligned} \left(\widehat{f_1^{g_1}} * \sigma_2\tau_2 \cdot \widehat{f_2^{g_2}}\right)(i) &= \widehat{f_1^{g_1}}(\sigma_2\tau_2(i)) \cdot \widehat{f_2^{g_2}}(i) \\ &= f_1(\sigma_2(i)) \cdot f_2(i) \end{aligned}$$

because $\tau_2(i) = i$ for all $i \leq k-1$. Finally, for $k \leq i \leq k+1$, we have

$$\widehat{(f_1 * \sigma_2 \cdot f_2)^{g_1 g_2}}(i) = (f_1 * \sigma_2 \cdot f_2)^{g_1 g_2}(k) = g_1 g_2$$

and

$$\left(\widehat{f_1^{g_1}} * \sigma_2\tau_2 \cdot \widehat{f_2^{g_2}}\right)(i) = \widehat{f_1^{g_1}}(\sigma_2\tau_2(i)) \cdot \widehat{f_2^{g_2}}(i) = g_1 g_2,$$

where the last equality follows because σ_2 fixes $\{k, k+1\}$ and $\widehat{f_j^{g_j}}$ takes the value g_j on both k and $k+1$. This completes the proof. $\square$

4.6 The quiver computation

Let $U \in \operatorname{IRep} G \wr S_k$ and $V \in \operatorname{IRep} G \wr S_{k+1}$. To simplify the notation, we shall use the same notation for a module and its character. For example, we write U and V in place of χ_U and χ_V.

The number of arrows in the quiver of $\mathbb{C}(G \wr \mathrm{SE}_n)$ from V to U is the multiplicity of $U \otimes V^*$ as a simple module in the $(G \wr S_k \times G \wr S_{k+1})$-module

$$\mathbb{C}X = \operatorname{Ind}_K^{G\wr S_k \times G\wr S_{k+1}} \operatorname{tr}_K .$$

This multiplicity can be expressed as the inner product of characters:

$$\langle U \otimes V^*, \operatorname{Ind}_K^{G\wr S_k \times G\wr S_{k+1}} \operatorname{tr}_K \rangle.$$

Using Frobenius reciprocity, this equals

$$\langle \operatorname{Res}_K^{G\wr S_k \times G\wr S_{k+1}} (U \otimes V^*), \operatorname{tr}_K \rangle.$$

According to Lemma 4.8, a general element of K is of the form

$$((f^g, \sigma), (\hat{f}^g, \sigma\tau))$$

where $\sigma \in S_{k-1}$, $\tau \in S_{\{k,k+1\}}$, $f \in G^{[k-1]}$, $g \in G$. Therefore,

$$\begin{aligned} &\langle \operatorname{Res}_K^{G\wr S_k \times G\wr S_{k+1}} (U \otimes V^*), \operatorname{tr}_K \rangle \\ &\quad = \frac{1}{|K|} \sum_{\substack{f \in G^{[k-1]}, g \in G \\ \sigma \in S_{k-1}, \tau \in S_2}} U \otimes V^*(((f^g, \sigma), (\hat{f}^g, \sigma\tau))) \operatorname{tr}_K(((f^g, \sigma), (\hat{f}^g, \sigma\tau))) \\ &\quad = \frac{1}{|K|} \sum_{\substack{f \in G^{[k-1]}, g \in G \\ \sigma \in S_{k-1}, \tau \in S_2}} U((f^g, \sigma)) \cdot V^*((\hat{f}^g, \sigma\tau)) \\ &\quad = \frac{1}{|K|} \sum_{\substack{f \in G^{[k-1]}, g \in G \\ \sigma \in S_{k-1}, \tau \in S_2}} U((f^g, \sigma)) \cdot \overline{V((\hat{f}^g, \sigma\tau))}. \end{aligned}$$

Now, if we think of $(\hat{f}^g, \sigma\tau)$ as a $(k-1)\times(k-1)$ and 2×2 block matrix, it is just a general element of $G \wr S_{k-1} \times G \times S_2$ so we can write

$$V((\hat{f}^g, \sigma\tau)) = \mathrm{Res}^{G\wr S_{k+1}}_{G\wr S_{k-1}\times(G\times S_2)} V((\hat{f}^g, \sigma\tau))$$

Likewise, if we view (f^g, σ) as a $(k-1)\times(k-1)$ and 1×1 block matrix, it is a general element of $G \wr S_{k-1} \times G$ so

$$U((f^g, \sigma)) = \mathrm{Res}^{G\wr S_k}_{G\wr S_{k-1}\times G} U((f^g, \sigma)).$$

In order to view this also as a $K = (G \wr S_{k-1} \times (G \times S_2))$ module we will write this as

$$\mathrm{Res}^{G\wr S_k}_{G\wr S_{k-1}\times G} U((f^g, \sigma)) \cdot \mathrm{tr}_2(\tau).$$

where tr_2 is the trivial module of S_2. Therefore, we obtain

$$\begin{aligned}
&\langle \mathrm{Res}^{G\wr S_k \times G\wr S_{k+1}}_{K} (U \otimes V^*), \mathrm{tr}_K\rangle \\
&\quad = \frac{1}{|K|} \sum_{\substack{f\in G^{[k-1]}, g\in G\\ \sigma\in S_{k-1}, \tau\in S_2}} U((f^g,\sigma)) \cdot \overline{V((\hat{f}^g, \sigma\tau))} \\
&\quad = \frac{1}{|K|} \sum_{\substack{f\in G^{[k-1]}, g\in G\\ \sigma\in S_{k-1}, \tau\in S_2}} \Big(\mathrm{Res}^{G\wr S_k}_{G\wr S_{k-1}\times G} U((f^g,\sigma)) \cdot \mathrm{tr}_2(\tau)\Big) \cdot \overline{\mathrm{Res}^{G\wr S_{k+1}}_{G\wr S_{k-1}\times(G\times S_2)} V((\hat{f}^g, \sigma\tau))} \\
&\quad = \frac{1}{|K|} \sum_{\substack{((f,\sigma),(g,\tau))\\ \in (G\wr S_{k-1})\times(G\times S_2)}} ((\mathrm{Res}^{G\wr S_k}_{G\wr S_{k-1}\times G} U) \boxtimes \mathrm{tr}_2)((f^g,\sigma,\tau)) \cdot \overline{\mathrm{Res}^{G\wr S_{k+1}}_{G\wr S_{k-1}\times(G\times S_2)} V((\hat{f}^g, \sigma\tau))} \\
&\quad = \langle (\mathrm{Res}^{G\wr S_k}_{G\wr S_{k-1}\times G} U) \boxtimes \mathrm{tr}_2, \mathrm{Res}^{G\wr S_{k+1}}_{G\wr S_{k-1}\times(G\times S_2)} V\rangle.
\end{aligned}$$

By Frobenius reciprocity, this equals

$$\langle \mathrm{Ind}^{G\wr S_{k+1}}_{G\wr S_{k-1}\times(G\times S_2)} ((\mathrm{Res}^{G\wr S_k}_{G\wr S_{k-1}\times G} U) \boxtimes \mathrm{tr}_2), V\rangle.$$

By transitivity of induction we obtain

$$\langle \mathrm{Ind}^{G\wr S_{k+1}}_{G\wr S_{k-1}\times(G\wr S_2)} \mathrm{Ind}^{G\wr S_{k-1}\times(G\wr S_2)}_{G\wr S_{k-1}\times(G\times S_2)} ((\mathrm{Res}^{G\wr S_k}_{G\wr S_{k-1}\times G} U) \boxtimes \mathrm{tr}_2), V\rangle.$$

Now we analyze this expression using the branching rules described in Section 3. Let U be a simple module of $G \wr S_k$ corresponding to a multipartition Λ.

By Proposition 3.4, we have

$$\mathrm{Res}^{G\wr S_{k+1}}_{(G\wr S_k)\times G}(S^\Lambda) = \bigoplus_{r=1}^{l} \bigoplus_{\Gamma\in Y_r^-(\Lambda)} S^\Gamma \boxtimes U_r,$$

where $\Gamma \in Y_r^-(\Lambda)$ means that Γ is obtained by removing one box from the r-th component of Λ. Let $S^\Gamma \boxtimes U_r$ be one component in this summation. The next step is to compute the induction

$$\mathrm{Ind}^{G\wr S_{k-1}\times(G\wr S_2)}_{G\wr S_{k-1}\times(G\times S_2)}(S^\Gamma \boxtimes U_r \boxtimes \mathrm{tr}_2)$$

For this we need to compute the induction

$$\operatorname{Ind}_{(G\times S_2)}^{G\wr S_2}(U_r \boxtimes \operatorname{tr}_2).$$

Let $m_{i,j}^{r}$ be the multiplicity of U_r in $U_i \otimes U_j$, let $m_i^{+,r}$ be the multiplicity of U_r in $\operatorname{Sym}^2 U_i$, and let $m_i^{-,r}$ be the multiplicity of U_r in $\operatorname{Alt}^2 U_i$. According to Lemma 3.2 and Lemma 3.3 we have that

$$\operatorname{Ind}_{(G\times S_2)}^{G\wr S_2}(U_r \boxtimes \operatorname{tr}_2) = \bigoplus_{1\leq i<j\leq l} \left(m_{i,j}^{r} W_{i,j}\right) \oplus \bigoplus_{i=1}^{l} \left(m_i^{+,r} W_i^{+} \oplus m_i^{-,r} W_i^{-}\right)$$

so

$$\operatorname{Ind}_{G\wr S_{k-1}\times(G\times S_2)}^{G\wr S_{k-1}\times(G\wr S_2)} \left(S^{\Gamma}\boxtimes U_r \boxtimes \operatorname{tr}_2\right) = \bigoplus_{1\leq i<j\leq l} \left(m_{i,j}^{r}(S^{\Gamma}\boxtimes W_{i,j})\right) \oplus \bigoplus_{i=1}^{l} \left(m_i^{+,r}(S^{\Gamma}\boxtimes W_i^{+}) \oplus m_i^{-,r}(S^{\Gamma}\boxtimes W_i^{-})\right).$$

Finally, by Proposition 3.5 we have that

$$\operatorname{Ind}_{G\wr S_{k-1}\times(G\wr S_2)}^{G\wr S_{k+1}} \left(\bigoplus_{1\leq i<j\leq l} m_{i,j}^{r}(S^{\Gamma}\boxtimes W_{i,j}) \oplus \bigoplus_{i=1}^{l} \left(m_i^{+,r}(S^{\Gamma}\boxtimes W_i^{+}) \oplus m_i^{-,r}(S^{\Gamma}\boxtimes W_i^{-})\right)\right) =$$

$$\bigoplus_{1\leq i<j\leq l} \left(m_{i,j}^{r} \bigoplus_{\Delta\in Y_{i,j}^{+}(\Gamma)} S^{\Delta}\right) \oplus \bigoplus_{i=1}^{l} \left(m_i^{+,r} \bigoplus_{\Delta\in Y_{i,H^2}^{+}(\Gamma)} S^{\Delta} \oplus m_i^{-,r} \bigoplus_{\Delta\in Y_{i,V^2}^{+}(\Gamma)} S^{\Delta}.\right)$$

From this we can conclude:

Theorem 4.10. *Let G be a group with l conjugacy classes and let $\operatorname{IRep} G = \{U_1, \ldots, U_l\}$. Define $m_{i,j}^{r}, m_i^{+,r}, m_i^{-,r}$ as above. The quiver Q of the monoid $G\wr\mathrm{PT}_n$ is described as follows. The vertices correspond to multi-Young diagrams with l components and k boxes, where k varies from 0 to n. Let U be a multi-Young diagram with k boxes, and V be a multi-Young diagram with p boxes. If $p \neq k+1$, then there are no arrows from V to U. If $p = k+1$, then the number of arrows depends on the ways that V can be constructed from U by removing one box and adding two. For each way that V can be constructed from U by removing one box from the r-th component and adding one box in components i and j ($i \neq j$) we add $m_{i,j}^{r}$ arrows. In each way that V can be constructed from U by removing one box from the r-th component and adding two boxes on the i-th component but not on the same column we add $m_i^{+,r}$ arrows. In each way that V can be constructed from U by removing one box from the r-th component and adding two boxes on the i-th component but not on the same row we add $m_i^{-,r}$ arrows.*

4.7 Examples

4.7.1 The quiver of $\mathbb{C}\,\mathrm{PT}_n$

If G is the trivial group then $G\wr\mathrm{PT}_n \simeq \mathrm{PT}_n$. In this case, $\operatorname{IRep} G$ contains only the trivial module $U_1 = \operatorname{tr}_G$. Clearly $\operatorname{Sym}^2 \operatorname{tr}_G = \operatorname{tr}_G$ and $\operatorname{Alt}^2 \operatorname{tr}_G = 0$. Therefore, $m_1^{+,1} = 1$ and $m_1^{-,1} = 0$. Therefore, we retrieve [26, Theorem 3.8]:

Theorem 4.11. *The vertices of the quiver Q of the monoid algebra $\mathbb{C}\mathrm{PT}_n$ correspond to Young diagrams with k boxes where k varies from 0 to n. Let U be a Young diagram with k boxes and V be a Young diagram with p boxes. If $p \neq k+1$, then there are no arrows from V to U. If $p = k+1$, then the number of arrows from V to U is the number of ways that V can be constructed from U by removing one box and adding two, but not in the same column.*

4.7.2 Generalized monoid of partial functions

Let $G = C_l$ be the cyclic group with l elements, say $G = \{1_G, g, g^2, \ldots, g^{l-1}\}$ where $g^l = 1_G$. All simple modules $\rho \in \mathrm{IRep}\, G$ are one-dimensional. There are l such modules, ρ_r for $r = 0, \ldots, l-1$, defined by:

$$\rho_r(g^n) = \omega^{rn}$$

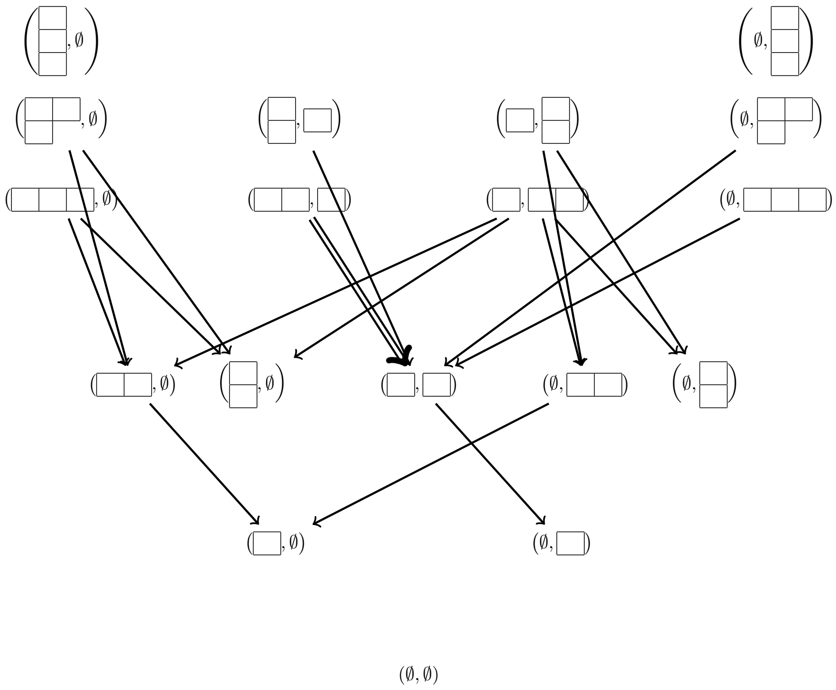
where $\omega = e^{2\pi i/l}$ is a primitive l-th root of unity. These modules satisfy $\rho_i \otimes \rho_j = \rho_{i+j}$, where the sum is taken modulo l. Since each ρ_r is one-dimensional, its symmetric square and alternating square satisfy:

$$\mathrm{Sym}^2\rho_r = \rho_r \otimes \rho_r = \rho_{2r} \quad \text{and} \quad \mathrm{Alt}^2\rho_r = 0$$

Therefore, the decomposition coefficients $m^r_{i,j}$ for the tensor product are given by $m^r_{i,j} = 1$ if $i + j \equiv r \pmod{l}$ and zero otherwise. For the symmetric and alternating squares, the coefficients are $m_i^{+,r} = 1$ if $2i \equiv r \pmod{l}$ and zero otherwise, while $m_i^{-,r} = 0$ for all $r \in \{0, \ldots, l-1\}$. We end with the following result:

Theorem 4.12. *The vertices of the quiver Q of the monoid algebra $\mathbb{C}(C_l \wr \mathrm{PT}_n)$ correspond to mutli-Young diagrams with l components, indexed $\{0, \ldots l-1\}$, and k boxes, where k varies from 0 to n. Let U be a multi-Young diagram with k boxes and V be a multi-Young diagram with p boxes. If $p \neq k+1$, then there are no arrows from V to U. If $p = k+1$, then the number of arrows depends on the ways that V can be constructed from U by removing one box and adding two. For each way that V can be constructed from U by removing one box from the r-th component and adding one box in components $i$and j we add an arrow if $i + j \equiv r \pmod{l}$. In each way that V can be constructed from U by removing one box from the r-th component and adding two boxes on the i-th component but not on the same column we add an arrow if $2i \equiv r \pmod{l}$.*

Example 4.13. Consider the case $G = C_2$. The quiver of the algebra $\mathbb{C}(C_2 \wr \mathrm{PT}_3)$ is given by the following figure:



The first component of each multi-Young diagram corresponds to the trivial module ρ_0, while the second component corresponds to the alternating module ρ_1.

Notably, there are two arrows from $(\square\square, \square)$ to $(\square, \square)$. This multiplicity arises because the branching construction can be satisfied in two distinct ways:

- First, by removing one box from the first component ($r = 0$) and adding two boxes in the same row, which is valid since $0 + 0 \equiv 0 \pmod 2$.
- Second, by removing one box from the second component ($r = 1$) and adding one box to each component, which is valid since $0 + 1 \equiv 1 \pmod 2$.

4.7.3 The quiver of $\mathbb{C}(S_3 \wr \mathrm{PT}_n)$

We now consider the case of the smallest non-abelian group $G = S_3$. The set $\mathrm{IRep}(G)$ contains three simple modules: The trivial module U_1, the sign module U_2, and the standard module U_3. It is well-known and easily verified by character considerations that the symmetric and alternating squares satisfy:

$$\begin{aligned} \mathrm{Sym}^2 U_1 &= \mathrm{Sym}^2 U_2 = U_1, & \mathrm{Sym}^2 U_3 &= U_3 \oplus U_1, \\ \mathrm{Alt}^2 U_1 &= \mathrm{Alt}^2 U_2 = 0, & \mathrm{Alt}^2 U_3 &= U_2. \end{aligned}$$

Furthermore, the tensor products are given by:

$$U_1 \otimes U_2 = U_2, \quad U_1 \otimes U_3 = U_3, \quad U_2 \otimes U_3 = U_3.$$

Therefore, the coefficients $m^r_{i,j}$ are given by:

$$m^r_{1,2} = \begin{cases} 1 & \text{if } r = 2, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad m^r_{1,3} = m^r_{2,3} = \begin{cases} 1 & \text{if } r = 3, \\ 0 & \text{otherwise.} \end{cases}$$

The coefficients $m^{q,r}_1, m^{q,r}_2$, where $q \in \{+,-\}$, satisfy:

$$m^{q,r}_1 = m^{q,r}_2 = \begin{cases} 1 & \text{if } q = + \text{ and } r = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for the coefficients $m^{q,r}_3$, where $q \in \{+,-\}$, we have:

$$m^{+,r}_3 = \begin{cases} 1 & \text{if } r \in \{1,3\}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad m^{-,r}_3 = \begin{cases} 1 & \text{if } r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

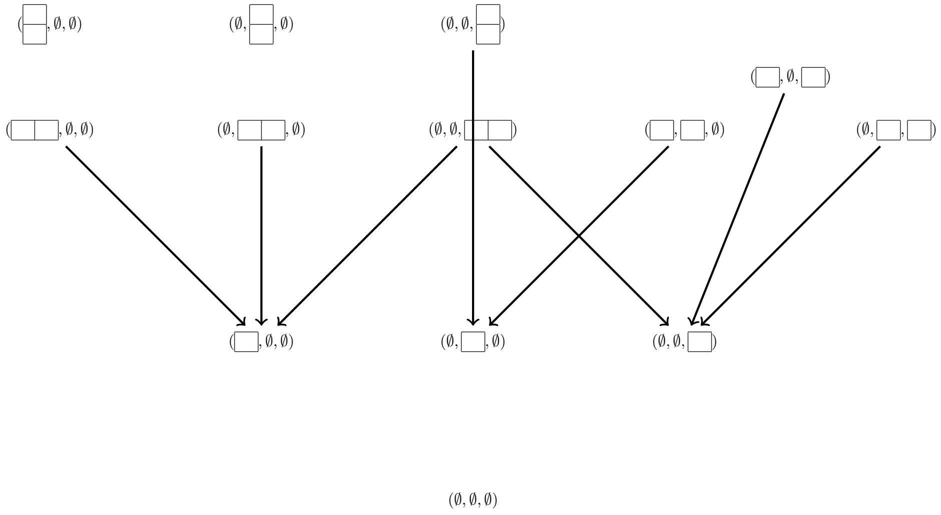
Therefore, we obtain:

Theorem 4.14. *The vertices of the quiver Q of the monoid algebra $\mathbb{C}(S_3 \wr \mathrm{PT}_n)$ correspond to multi-Young diagrams $\mathbf{\Lambda} = (\lambda_1, \lambda_2, \lambda_3)$ with k total boxes, where $0 \leq k \leq n$. Let U be a multi-Young diagram with k boxes and V be a multi-Young diagram with p boxes.*

If $p \neq k + 1$, there are no arrows from V to U. If $p = k + 1$, the number of arrows from V to U is the number of ways V can be constructed from U by removing one box and adding two boxes according to the following rules:

- *Remove one box from λ_1 and add two boxes to any one component, provided they are not in the same column.*
- *Remove one box from λ_2 and add one box to λ_1 and one box to λ_2.*
- *Remove one box from λ_2 and add two boxes to λ_3, provided they are not in the same row.*
- *Remove one box from λ_3 and add one box to λ_1 or λ_2 and another box to λ_3.*
- *Remove one box from λ_3 and add two boxes to λ_3, provided they are not in the same column.*

Example 4.15. The quiver of the algebra $\mathbb{C}(S_3 \wr \mathrm{PT}_2)$ is given by the following figure:



5 Global dimension of $\mathbb{C}(G \wr \mathrm{PT}_n)$

In this section, we prove that the global dimension of the algebra $\mathbb{C}(G \wr \mathrm{PT}_n)$ is $n-1$ based on the results established for the algebra $\mathbb{C}\,\mathrm{PT}_n$ in [29]. We show that every projective module of $\mathbb{C}\,\mathrm{PT}_n$ can be lifted to a corresponding projective module of $\mathbb{C}(G \wr \mathrm{PT}_n)$, allowing us to transfer the known homological properties to the wreath product case.

5.1 The global dimension of an algebra

Let A be a finite-dimensional $\mathbb{C}$-algebra. An A-module P is *projective* if the functor $\mathrm{Hom}_A(P, -)$ is exact, or equivalently, if P is a direct summand of a free module. A module is *indecomposable* if it cannot be written as a direct sum of two non-zero submodules. Every projective A-module decomposes into a direct sum of indecomposable projective modules.

These modules are classified by the idempotents of the algebra. Two idempotents e, f are *orthogonal* if $ef = fe = 0$. An idempotent is *primitive* if it cannot be written as a sum of two non-zero orthogonal idempotents. A set $\{e_1, \ldots, e_n\}$ is a *complete set of orthogonal primitive idempotents* if the elements are pairwise orthogonal, each e_i is primitive, and $\sum e_i = 1_A$. Every indecomposable projective A-module is isomorphic to Ae for some primitive idempotent e, and a complete set of primitive orthogonal idempotents yields all indecomposable projectives up to isomorphism.

Let A be a finite-dimensional $\mathbb{C}$-algebra. The *Jacobson radical* of A, denoted $\mathrm{Rad}(A)$, is defined as the intersection of all maximal left ideals of A. Equivalently, $\mathrm{Rad}(A)$ is the unique maximal nilpotent two-sided ideal of A. For any finite-dimensional A-module M, we define its *radical*, $\mathrm{Rad}(M)$, as the intersection of all maximal submodules of M. A fundamental property of the radical is that it can be computed via the action of the algebra's radical. Specifically, we have the identity $\mathrm{Rad}(M) = \mathrm{Rad}(A)M$.

Let A be a finite-dimensional $\mathbb{C}$-algebra and M an A-module. A *projective cover* of M is a pair (P, π), where P is a projective A-module and $\pi : P \to M$ is a surjection such that $\ker(\pi) \subseteq \mathrm{Rad}(P)$.

For the algebras considered here, every finite-dimensional module M admits a projective cover, which is unique up to isomorphism.

Let A be a finite-dimensional $\mathbb{C}$-algebra and M an A-module. A *projective resolution* of M is an exact sequence of the form

$$\mathbf{P}_\bullet : \cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

where each P_i is a projective A-module. Such a resolution is called *minimal* if P_0 is the projective cover of M, and for each $i \geq 1$, P_i is the projective cover of $\ker(d_{i-1})$.

Every finite-dimensional module M possesses a minimal projective resolution, which is unique up to isomorphism of complexes. The *projective dimension* of M, denoted $\mathrm{pd}(M)$, is the length of its minimal projective resolution, that is, the largest integer n such that $P_n \neq 0$ (or infinity if the resolution does not terminate). For a finite-dimensional algebra A, the *global dimension* of A, denoted $\mathrm{glDim}(A)$, is the supremum of the projective dimensions of all simple A-modules:

$$\mathrm{glDim}(A) = \sup\{\mathrm{pd}(S) \mid S \text{ is a simple } A\text{-module}\}$$

The global dimension is an invariant of the Morita equivalence class of the algebra A. If two algebras are Morita equivalent, their global dimensions are equal. Moreover, it is a known fact for finite-dimensional algebras that the global dimension is bounded above by the length of the longest path in the quiver of the algebra.

5.2 Lifting projective resolutions of EI-categories

Let E be a category and G be a finite group. For simplicity, assume that E is a subcategory of **Set** so that there is a natural wreath product $G \wr \mathrm{E}$. Clearly, every $\mathbb{C}\,\mathrm{E}$-module M can be inflated to a $\mathbb{C}(G \wr \mathrm{E})$-module via the action $(f, m) \bullet v = m \bullet v$. We denote the inflation of M to $\mathbb{C}(G \wr \mathrm{E})$ by $\mathrm{Inf}(M)$. In fact, we consider Inf as a functor between the category of $\mathbb{C}\,\mathrm{E}$-modules and $\mathbb{C}(G \wr \mathrm{E})$-modules. In this subsection, we show that if E is an EI-category the inflation functor lifts the minimal projective resolution of M to the minimal projective resolution of $\mathrm{Inf}(M)$.

Let E be an EI-category. For every object $e \in \mathrm{E}^0$, we denote by H_e the associated endomorphism group, and let P_e be a complete set of orthogonal primitive idempotents of $\mathbb{C}H_e$. It is known (see [12, Lemma 9.31] or [33, Corollary 4.5]) that we can obtain a complete set of orthogonal primitive idempotents P of $\mathbb{C}\,\mathrm{E}$ by taking the union of all complete sets of orthogonal primitive idempotents of the endomorphism groups:

$$P = \bigcup_{e \in \mathrm{E}^0} P_e.$$

Therefore, if V is a simple $\mathbb{C}H_e$-module isomorphic to $\mathbb{C}H_e p$ for some $p \in P_e$, then

$$\mathbb{C}\,\mathrm{E}\,p \cong \mathbb{C}\,\mathrm{E}\,e \otimes_{\mathbb{C}H_e} \mathbb{C}H_e p \cong \mathbb{C}\,\mathrm{E}\,e \otimes_{\mathbb{C}H_e} V$$

is an indecomposable projective module of $\mathbb{C}\,\mathrm{E}$. Moreover, every indecomposable projective module is isomorphic to $\mathbb{C}\,\mathrm{E}\,e \otimes_{\mathbb{C}H_e} V$ for some $e \in \mathrm{E}^0$ and $V \in \mathrm{IRep}(\mathbb{C}H_e)$.

It is straightforward to verify that $G \wr \mathrm{E}$ is also an EI-category with the same set of objects E^0. For any object $e \in \mathrm{E}^0$, the associated endomorphism group in the wreath product category is $G \wr H_e$. Given a simple module $V \in \mathrm{IRep}(\mathbb{C}H_e)$, its inflation $\mathrm{Inf}(V)$ is a module over $\mathbb{C}(G \wr H_e)$, which remains simple under the action defined previously. The relationship between the inflation functor and the indecomposable projectives is captured by the following lemma.

Lemma 5.1. *Let* $e \in \mathrm{E}^0$ *and* $V \in \mathrm{IRep}(\mathbb{C}H_e)$. *Then there is an isomorphism of* $\mathbb{C}(G \wr \mathrm{E})$*-modules:*

$$\mathrm{Inf}(\mathbb{C}\,\mathrm{E}\,e \otimes_{\mathbb{C}H_e} V) \simeq \mathbb{C}(G \wr \mathrm{E})e \otimes_{\mathbb{C}(G \wr H_e)} \mathrm{Inf}(V).$$

Proof. We denote by $\mathbf{1}_e$ the function in G^e that maps every element of the set e to the identity 1_G and by id_e we denote the identity function of e. To establish the isomorphism

$$\mathrm{Inf}(\mathbb{C}\,\mathrm{E}\,e \otimes_{\mathbb{C}H_e} V) \cong \mathbb{C}(G \wr \mathrm{E})e \otimes_{\mathbb{C}(G \wr H_e)} \mathrm{Inf}(V),$$

we first observe that for every $(f, m) \in (G \wr \mathrm{E})e$ and $v \in V$ we have

$$\begin{aligned}(f, m) \otimes v &= ((\mathbf{1}_e, m) \cdot (f, \mathrm{id}_e)) \otimes v \\ &= (\mathbf{1}_e, m) \otimes (f, \mathrm{id}_e) \bullet v \\ &= (\mathbf{1}_e, m) \otimes (\mathrm{id}_e \bullet v) \\ &= (\mathbf{1}_e, m) \otimes v,\end{aligned}$$

Next, define a linear map

$$\Psi : \mathrm{Inf}(\mathbb{C}\,\mathrm{E}\,e \otimes_{\mathbb{C}H_e} V) \to \mathbb{C}(G \wr \mathrm{E})e \otimes_{\mathbb{C}(G \wr H_e)} \mathrm{Inf}(V).$$

on simple tensors by

$$\Psi(m \otimes v) = (\mathbf{1}_e, m) \otimes v,$$

where $m \in \mathbb{C}\,\mathrm{E}\,e$ and $v \in V$. To see that Ψ is well-defined, note that for any $h \in H_e$, the inflation action on V satisfies $(\mathbf{1}_e, h) \bullet v = h \bullet v$. Thus,

$$\begin{aligned}\Psi(mh \otimes v) &= (\mathbf{1}_e, mh) \otimes v \\ &= (\mathbf{1}_e, m)(\mathbf{1}_e, h) \otimes v \\ &= (\mathbf{1}_e, m) \otimes (\mathbf{1}_e, h) \bullet v \\ &= (\mathbf{1}_e, m) \otimes h \bullet v \\ &= \Psi(m \otimes h \bullet v),\end{aligned}$$

which confirms Ψ respects the tensor relation over $\mathbb{C}H_e$.

To verify that Ψ is a $\mathbb{C}(G \wr \mathrm{E})$-module homomorphism, let $(f, m') \in G \wr \mathrm{E}$ where $m' \in \mathrm{E}(e', e'')$ and let $m \otimes v \in \mathbb{C}\,\mathrm{E}\,e \otimes_{\mathbb{C}H_e} V$ where $m \in \mathrm{E}(e, e')$. By the definition of the inflation action, we have:

$$\Psi((f, m') \bullet (m \otimes v)) = \Psi(m'm \otimes v) = (\mathbf{1}_e, m'm) \otimes v.$$

On the other hand, acting on the image gives:

$$\begin{aligned}(f, m') \bullet \Psi(m \otimes v) &= (f, m') \bullet ((\mathbf{1}_e, m) \otimes v) \\ &= ((f, m') \cdot (\mathbf{1}_e, m)) \otimes v \\ &= (f * m \cdot \mathbf{1}_e, m'm) \otimes v \\ &= (f * m, m'm) \otimes v \\ &= (\mathbf{1}_e, m'm) \otimes v.\end{aligned}$$

To prove that Ψ is an isomorphism, we construct its inverse

$$\Phi : \mathbb{C}(G \wr \mathrm{E})e \otimes_{\mathbb{C}(G \wr H_e)} \mathrm{Inf}(V) \to \mathrm{Inf}(\mathbb{C}\,\mathrm{E}\,e \otimes_{\mathbb{C}H_e} V).$$

Define Φ on spanning elements by $\Phi((f, m) \otimes v) = m \otimes v$. To show that Φ is well-defined, let $(g, h) \in G \wr H_e$; then:

$$\begin{aligned}\Phi((f, m) \cdot (g, h) \otimes v) &= \Phi((f * h \cdot g, mh) \otimes v) \\ &= mh \otimes v.\end{aligned}$$

On the other hand:

$$\begin{aligned}\Phi((f, m) \otimes (g, h) \bullet v) &= \Phi((f, m) \otimes h \bullet v) \\ &= m \otimes h \bullet v \\ &= mh \otimes v.\end{aligned}$$

Thus Φ is well-defined.

To show that Φ is a $\mathbb{C}(G \wr \mathrm{E})$-module homomorphism, let $(g, m') \in G \wr \mathrm{E}$ and $(f, m) \otimes v$ be a spanning element of the domain. We have:

$$\begin{aligned}\Phi((g, m') \bullet ((f, m) \otimes v)) &= \Phi(((g, m') \cdot (f, m)) \otimes v) \\ &= \Phi((g * m \cdot f, m'm) \otimes v) \\ &= m'm \otimes v.\end{aligned}$$

On the other hand, acting after applying Φ gives:

$$\begin{aligned}(g, m') \bullet \Phi((f, m) \otimes v) &= (g, m') \bullet (m \otimes v)\\ &= m'm \otimes v,\end{aligned}$$

so Φ is a $\mathbb{C}(G \wr \mathrm{E})$-module homomorphism.

From the definitions it follows immediately that $\Phi \circ \Psi$ is the identity on $\mathrm{Inf}(\mathbb{C}\,\mathrm{E}\, e \otimes_{\mathbb{C}H_e} V)$, as:

$$\Phi(\Psi(m \otimes v)) = \Phi((\mathbf{1}_e, m) \otimes v) = m \otimes v.$$

Conversely, for any $(f, m) \otimes v$ in the target, we previously established the identity $(f, m) \otimes v = (\mathbf{1}_e, m) \otimes v$. Therefore:

$$\Psi(\Phi((f, m) \otimes v)) = \Psi(m \otimes v) = (\mathbf{1}_e, m) \otimes v = (f, m) \otimes v,$$

which shows $\Psi \circ \Phi$ is the identity. Thus, Ψ is an isomorphism of $\mathbb{C}(G \wr \mathrm{E})$-modules. □

As a consequence of the fact that inflation preserves the structure of indecomposable projective modules, we obtain the following result regarding general projective modules.

Corollary 5.2. *If P is a projective $\mathbb{C}\,\mathrm{E}$-module, then its inflation $\mathrm{Inf}(P)$ is a projective $\mathbb{C}(G \wr \mathrm{E})$-module.*

Proof. Each projective $\mathbb{C}\,\mathrm{E}$-module P can be decomposed into a direct sum of indecomposable projectives, each isomorphic to $\mathbb{C}\,\mathrm{E}\, e \otimes_{\mathbb{C}H_e} V$ for some $e \in \mathrm{E}^0$ and $V \in \mathrm{IRep}(\mathbb{C}H_e)$. According to Lemma 5.1, the inflation of each such indecomposable projective is an indecomposable projective $\mathbb{C}(G \wr \mathrm{E})$-module. Since the inflation functor Inf preserves direct sums, it follows that $\mathrm{Inf}(P)$ is a direct sum of projective modules and is, therefore, projective. □

The next step is to characterize the radical of an EI-category algebra.

Proposition 5.3 ([11, Proposition 4.6])**.** *Let E be a finite EI-category. The radical $\mathrm{Rad}(\mathbb{C}\,\mathrm{E})$ is the subspace spanned by all non-invertible morphisms in E.*

In an EI-category, a morphism $m \in \mathrm{E}(e, e')$ is non-invertible if and only if its domain and codomain are not isomorphic ($e \not\simeq e'$).

Lemma 5.4. *Let P be a projective $\mathbb{C}\,\mathrm{E}$-module. Then $\mathrm{Inf}(\mathrm{Rad}\, P) = \mathrm{Rad}(\mathrm{Inf}\, P)$.*

Proof. First, assume P is an indecomposable projective module. Then $P \simeq \mathbb{C}\,\mathrm{E}\, e \otimes_{\mathbb{C}H_e} V$ for some $e \in \mathrm{E}^0$ and $V \in \mathrm{IRep}(\mathbb{C}H_e)$. The radical of P is given by $\mathrm{Rad}\, P = \mathrm{Rad}(\mathbb{C}\,\mathrm{E})P = \mathrm{Rad}(\mathbb{C}\,\mathrm{E})e \otimes_{\mathbb{C}H_e} V$. Using the characterization of the radical for EI-category algebras, $\mathrm{Rad}\, P$ and $\mathrm{Inf}(\mathrm{Rad}\, P)$ are spanned by the set

$$S = \{m \otimes v \mid m \in \mathrm{E}(e, e'),\ e' \not\simeq e,\ v \in V\}.$$

On the other hand, by Lemma 5.1, $\mathrm{Inf}(P) \simeq \mathbb{C}(G \wr \mathrm{E})e \otimes_{\mathbb{C}(G \wr H_e)} \mathrm{Inf}\, V$. The radical of the latter is $\mathrm{Rad}(\mathbb{C}(G \wr \mathrm{E}))e \otimes_{\mathbb{C}(G \wr H_e)} \mathrm{Inf}\, V$, which is spanned by simple tensors of the form $(f, m) \otimes v$ where $m \in \mathrm{E}(e, e')$ and $e' \not\simeq e$. As shown in the proof of Lemma 5.1, $(f, m) \otimes v = (\mathbf{1}_e, m) \otimes v$.

Under the isomorphism $\Phi : \mathbb{C}(G \wr \mathrm{E})e \otimes_{\mathbb{C}(G \wr H_e)} \mathrm{Inf}\, V \to \mathrm{Inf}\, P$, the image of the spanning set

$$\{(\mathbf{1}_e, m) \otimes v \mid m \in \mathrm{E}(e, e'),\ e \not\simeq e',\ v \in V\}$$

is exactly the set S defined above. Therefore, S also spans $\operatorname{Rad}(\operatorname{Inf} P)$. Since both $\operatorname{Inf}(\operatorname{Rad} P)$ and $\operatorname{Rad}(\operatorname{Inf} P)$ are spanned by the same set of elements within $\operatorname{Inf} P$, we have $\operatorname{Rad}(\operatorname{Inf} P) = \operatorname{Inf}(\operatorname{Rad} P)$.

The result for a general projective module P follows immediately from the fact that P is a direct sum of indecomposable projectives and both the radical and the inflation functor preserve direct sums. ☐

Lemma 5.5. *Let M be a $\mathbb{C}\,\mathrm{E}$-module and let $P \xrightarrow{\pi} M$ be its projective cover. Then* $\operatorname{Inf}(P) \xrightarrow{\operatorname{Inf}(\pi)} \operatorname{Inf}(M)$ *is the projective cover of the $\mathbb{C}(G \wr \mathrm{E})$-module* $\operatorname{Inf}(M)$.

Proof. Recall that a surjective homomorphism $\pi : P \to M$ is a projective cover if and only if P is a projective module and $\ker(\pi) \subseteq \operatorname{Rad}(P)$.

Applying Inf yields

$$\operatorname{Inf}(P) \xrightarrow{\operatorname{Inf}(\pi)} \operatorname{Inf}(M),$$

and it is clear that $\operatorname{Inf}(\pi)$ is surjective and $\ker(\operatorname{Inf}(\pi)) \simeq \operatorname{Inf}(\ker(\pi))$.

Next, we know from Lemma 5.1 that $\operatorname{Inf}(P)$ is a projective $\mathbb{C}(G \wr \mathrm{E})$-module. Finally, applying the result of our previous lemma, we have:

$$\ker(\operatorname{Inf}(\pi)) \simeq \operatorname{Inf}(\ker(\pi)) \subseteq \operatorname{Inf}(\operatorname{Rad} P) = \operatorname{Rad}(\operatorname{Inf} P).$$

Since $\operatorname{Inf}(P)$ is projective and the kernel of the surjection is contained in its radical, $\operatorname{Inf}(P) \xrightarrow{\operatorname{Inf}(\pi)} \operatorname{Inf}(M)$ is the projective cover of $\operatorname{Inf}(M)$. ☐

Corollary 5.6. *Let $\mathbf{P}_\bullet \to M$ be a minimal projective resolution of a $\mathbb{C}\,\mathrm{E}$-module M. Then the inflated sequence* $\operatorname{Inf}(\mathbf{P}_\bullet) \to \operatorname{Inf}(M)$ *is a minimal projective resolution of the $\mathbb{C}(G \wr \mathrm{E})$-module* $\operatorname{Inf}(M)$.

Proof. Let $\mathbf{P}_\bullet$ be the resolution $\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$. It is easy to verify that Inf is an exact functor. Therefore, the sequence

$$\cdots \to \operatorname{Inf}(P_1) \xrightarrow{\operatorname{Inf}(d_1)} \operatorname{Inf}(P_0) \xrightarrow{\operatorname{Inf}(\epsilon)} \operatorname{Inf}(M) \to 0$$

is exact. By Lemma 5.1, each $\operatorname{Inf}(P_i)$ is a projective $\mathbb{C}(G \wr \mathrm{E})$-module.

By definition, the resolution $\mathbf{P}_\bullet$ is minimal if ϵ is a projective cover and each d_n induces a projective cover $P_n \to \ker(d_{n-1})$. By Lemma 5.5, inflation preserves the projective cover property. Thus, $\operatorname{Inf}(\epsilon)$ is the projective cover of $\operatorname{Inf}(M)$ and each $\operatorname{Inf}(d_n)$ induces the projective cover of $\operatorname{Inf}(\ker(d_{n-1})) \simeq \ker(\operatorname{Inf}(d_{n-1}))$. It follows that $\operatorname{Inf}(\mathbf{P}_\bullet)$ is a minimal projective resolution. ☐

Corollary 5.7. *The inflation functor preserves the projective dimension of modules. That is, for any $\mathbb{C}\,\mathrm{E}$-module M, we have*

$$\operatorname{pd}_{\mathbb{C}\,\mathrm{E}}(M) = \operatorname{pd}_{\mathbb{C}(G \wr \mathrm{E})}(\operatorname{Inf} M).$$

5.3 The Global Dimension

Finally, we apply the preceding results to compute the global dimension of $\mathbb{C}(G \wr \mathrm{PT}_n)$. Recall that E_n is the EI-category whose objects are the subsets of $[n] = \{1, \ldots, n\}$ and whose morphisms are onto functions. Since $\mathbb{C}(G \wr \mathrm{PT}_n)$ is Morita equivalent to the category algebra $\mathbb{C}(G \wr \mathrm{E}_n)$, we may instead determine the global dimension of the latter.

Theorem 5.8. *The global dimension of the category algebra* $\mathbb{C}(G \wr \mathrm{E}_n)$ *is* $n-1$*.*

Proof. We first establish $n-1$ as an upper bound. The global dimension is bounded above by the length of the longest path in the quiver of the algebra. In Theorem 4.10, we characterized the vertices of the quiver as multi-Young diagrams. Specifically, we showed that if U is a multi-Young diagram with k boxes and V is one with p boxes, an arrow $V \to U$ can only exist if $p = k+1$. Furthermore, there are no arrows directed toward the vertex corresponding to a multipartition with 0 boxes (the "bottom" element). Consequently, the longest directed path in the quiver has length $n-1$, which provides the required upper bound: $\mathrm{glDim}(\mathbb{C}(G \wr \mathrm{E}_n)) \leq n-1$.

For the lower bound, we consider the $\mathbb{C}\,\mathrm{E}_n$-simple module M corresponding to the partition $[2, 1^{n-2}]$. It was shown in [29, Corollary 6.9] that $\mathrm{pd}_{\mathbb{C}\,\mathrm{E}_n}(M) = n-1$. By Corollary 5.7, the inflation functor Inf preserves projective dimension, implying:

$$\mathrm{pd}_{\mathbb{C}(G \wr \mathrm{E}_n)}(\mathrm{Inf}\, M) = \mathrm{pd}_{\mathbb{C}\,\mathrm{E}_n}(M) = n-1.$$

Since the global dimension of an algebra is the supremum of the projective dimensions of its simple modules, it follows that $\mathrm{glDim}(\mathbb{C}(G \wr \mathrm{E}_n)) \geq n-1$. Combining these bounds, we conclude that the global dimension is exactly $n-1$. □

Corollary 5.9. *For every finite group* G*, the global dimension of the monoid algebra* $\mathbb{C}(G \wr \mathrm{PT}_n)$ *is* $n-1$*.*

6 The quiver of the wreath product of a group with the monoid of all order-preserving partial functions

A partial function $\alpha\colon [n] \to [n]$ is called *order-preserving* if $x \leq y \implies \alpha(x) \leq \alpha(y)$ for every x, y in the domain of α. Let PO_n be the submonoid of PT_n consisting of all order-preserving partial functions. In this section, we describe the quiver of the complex algebra $\mathbb{C}(G \wr \mathrm{PO}_n)$.

In this case, the wreath product is

$$G \wr \mathrm{PO}_n = \{(f, \alpha) \mid f \in \mathrm{PT}([n], G),\ \alpha \in \mathrm{PO}_n,\ \text{and}\ \mathrm{dom}(\alpha) = \mathrm{dom}(f)\}.$$

Recall the set of idempotents $\mathcal{E} = \{(1_X, \mathrm{id}_X) \mid X \subseteq [n]\}$, which forms a subsemilattice of $G \wr \mathrm{PO}_n$. It is a straightforward consequence that $G \wr \mathrm{PO}_n$ is an $\mathcal{E}$-Ehresmann and a right restriction monoid.

Let EO_n be the subcategory of the category E_n, defined in Section 4.1, which has the same set of objects but whose morphisms consist of surjective, total order-preserving functions. It follows that $G \wr \mathrm{EO}_n$ is the Ehresmann category associated with $G \wr \mathrm{PO}_n$. Thus, by Theorem 4.2, we obtain an isomorphism of algebras

$$\mathbb{C}(G \wr \mathrm{PO}_n) \simeq \mathbb{C}(G \wr \mathrm{EO}_n).$$

Following the approach in Section 4.2, we now consider the skeleton of $G \wr \mathrm{EO}_n$. If $\alpha\colon X \to Y$ is an order-preserving bijection, then its inverse α^{-1} is also order-preserving. Consequently, Lemma 4.4 implies that two objects X and Y of $G \wr \mathrm{EO}_n$ are isomorphic if and only if $|X| = |Y|$. Let SEO_n denote the category whose objects are $[k]$ for $0 \leq k \leq n$, and whose morphisms are surjective, total order-preserving functions. The skeleton of $G \wr \mathrm{EO}_n$ is then the wreath product $G \wr \mathrm{SEO}_n$. Our goal now is to describe the quiver of the algebra $\mathbb{C}(G \wr \mathrm{SEO}_n)$.

The endomorphism groups in $G \wr \mathrm{SEO}_n$ have a straightforward structure. Since the only order-preserving bijection from $[k]$ to itself is the identity map, the endomorphism group of any object $[k]$ in $G \wr \mathrm{SEO}_n$ is simply the direct product G^k.

By applying Theorem 4.5 to the skeletal category $G \wr \mathrm{SEO}_n$, the set of vertices of Q is the disjoint union of the simple modules of the endomorphism groups of the objects in the skeleton. Since the endomorphism group of the object $[k]$ is isomorphic to G^k, the vertex set is:

$$\bigsqcup_{k=0}^{n} \mathrm{IRep}(G^k).$$

Any simple module of G^k is an outer tensor product of k simple modules of G. Thus, the vertices of the quiver are naturally indexed by sequences $(V_1, \dots, V_k)$ of length $0 \le k \le n$, where each $V_i \in \mathrm{IRep}(G)$. So the vertex set is:

$$\bigsqcup_{k=0}^{n} \{V_1 \boxtimes \cdots \boxtimes V_k \mid V_i \in \mathrm{IRep}(G)\}.$$

For $k = 0$, the unique vertex is the trivial module of the trivial group G^0.

Our next step is to identify the irreducible morphisms in the category $G \wr \mathrm{SEO}_n$.

Lemma 6.1. *The irreducible morphisms of $G \wr \mathrm{SEO}_n$ are precisely the morphisms from $[k+1]$ to $[k]$ for $0 \le k < n$. That is,*

$$\mathrm{Irr}(G \wr \mathrm{SEO}_n)([r],[k]) = \begin{cases} G \wr \mathrm{SEO}_n([r],[k]) & \text{if } r = k+1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. As in the proof of Lemma 4.6, any morphism from $[k+1]$ to $[k]$ is clearly irreducible because any factorization would force one of the factors to be an isomorphism. Conversely, suppose $(f, \alpha) \in G \wr \mathrm{SEO}_n([r],[k])$ with $r > k+1$. Since α is a surjective order-preserving function, it is known that α factors as $\alpha = \alpha_1\alpha_2$, where $\alpha_2 \colon [r] \to [k+1]$ and $\alpha_1 \colon [k+1] \to [k]$ are both surjective order-preserving functions (see [26, Lemma 5.1]). We may then define the morphisms $(\mathbf{1}_{[k+1]}, \alpha_1)$ and (f, α_2). Since neither α_1 nor α_2 is a bijection, these factors are not invertible in $G \wr \mathrm{SEO}_n$. Their product is

$$(\mathbf{1}_{[k+1]}, \alpha_1) \cdot (f, \alpha_2) = (\mathbf{1}_{[k+1]} * \alpha_2 \cdot f, \alpha_1\alpha_2) = (f, \alpha),$$

which shows that (f, α) is not irreducible. □

Let $V \in \mathrm{IRep}(G^p)$ and $U \in \mathrm{IRep}(G^k)$ be vertices of the quiver. If $p \ne k+1$, there are no arrows in the quiver of $\mathbb{C}(G \wr \mathrm{SEO}_n)$ from V to U because $\mathrm{Irr}(G \wr \mathrm{SEO}_n)([p],[k])$ is empty. Consequently, we focus on the case $p = k+1$ and examine the structure of $\mathbb{C}[\mathrm{Irr}(G \wr \mathrm{SEO}_n)([k+1],[k])]$ as a $(G^k \times G^{k+1})$-module, with the action defined in Theorem 4.5.

A morphism in $\mathrm{Irr}(G \wr \mathrm{SEO}_n)([k+1],[k])$ is a pair (f, α), where $\alpha \colon [k+1] \to [k]$ is a surjective order-preserving function and $f \colon [k+1] \to G$. It is easy to see that there are exactly k surjective order-preserving maps from $[k+1]$ to $[k]$, denoted $\sigma_1, \dots, \sigma_k$. The map σ_i is defined by:

$$\sigma_i(j) = \begin{cases} j & \text{if } j \le i, \\ j-1 & \text{if } j > i. \end{cases}$$

so $\alpha = \sigma_i$ for some i. Set $M = \mathbb{C}[\mathrm{Irr}(G \wr \mathrm{SEO}_n)([k+1],[k])]$ and define

$$M_i = \mathrm{span}\{(f, \sigma_i) \mid f \in G^{[k+1]}\}.$$

We naturally identify $G^{[k]}$ with G^k.

Lemma 6.2. *The $(G^k \times G^{k+1})$-module M decomposes as a direct sum of k submodules:*

$$M \cong \bigoplus_{i=1}^{k} M_i,$$

Proof. Since the set of all such pairs (f, α) forms a basis for M, and since the M_i are disjoint except for the zero vector and their union spans M, we can write M as a direct sum of vector spaces

$$M \cong \bigoplus_{i=1}^{k} M_i.$$

To show this is a decomposition of $(G^k \times G^{k+1})$-modules, we examine the action defined in Theorem 4.5. For $h \in G^k$ and $g \in G^{k+1}$, the action on a basis element (f, σ_i) is:

$$(h, g) \bullet (f, \sigma_i) = (h, \mathrm{id}_{[k]}) \cdot (f, \sigma_i) \cdot (g, \mathrm{id}_{[k+1]})^{-1}.$$

Since $(g, \mathrm{id}_{[k+1]})^{-1} = (g^{-1}, \mathrm{id}_{[k+1]})$ and by calculating the product in the wreath product category:

$$(h, \mathrm{id}_{[k]}) \cdot (f, \sigma_i) \cdot (g^{-1}, \mathrm{id}_{[k+1]}) = ((h * \sigma_i) \cdot f \cdot g^{-1}, \sigma_i).$$

Crucially, the underlying order-preserving map σ_i remains unchanged by the action of the endomorphism groups so each subspace M_i is a submodule of M. □

For each $1 \leq i \leq k$, let $X_i = \{(f, \sigma_i) \mid f \in G^{[k+1]}\}$ denote the natural basis for the subspace M_i. By Lemma 6.2, the action of $G^k \times G^{k+1}$ preserves M_i and maps basis elements to basis elements. Consequently, we can view M_i as the permutation module $\mathbb{C}X_i$ arising from the action of $G^k \times G^{k+1}$ on the set X_i. To analyze the structure of this permutation module, we first establish that this action is transitive.

Lemma 6.3. *For each $1 \leq i \leq k$, the action of $G^k \times G^{k+1}$ on the set X_i is transitive.*

Proof. Let $(\mathbf{1}_{[k+1]}, \sigma_i)$ be the element of X_i where $\mathbf{1}_{[k+1]}$ is the constant function mapping every element of $[k+1]$ to 1_G. For any arbitrary $(f, \sigma_i) \in X_i$, we choose $(\mathbf{1}_{[k]}, f^{-1}) \in G^k \times G^{k+1}$. Applying the action, we have:

$$\begin{aligned}
(\mathbf{1}_{[k]}, f^{-1}) \bullet (\mathbf{1}_{[k+1]}, \sigma_i) &= (\mathbf{1}_{[k]}, \mathrm{id}_{[k]}) \cdot (\mathbf{1}_{[k+1]}, \sigma_i) \cdot (f^{-1}, \mathrm{id}_{[k+1]})^{-1} \\
&= (\mathbf{1}_{[k]}, \mathrm{id}_{[k]}) \cdot (\mathbf{1}_{[k+1]}, \sigma_i) \cdot (f, \mathrm{id}_{[k+1]}) \\
&= ((\mathbf{1}_{[k]} * \sigma_i) \cdot \mathbf{1}_{[k+1]} \cdot f, \sigma_i) \\
&= (\mathbf{1}_{[k+1]} \cdot \mathbf{1}_{[k+1]} \cdot f, \sigma_i) \\
&= (f, \sigma_i).
\end{aligned}$$

This shows that any element of X_i can be reached from $(\mathbf{1}_{[k+1]}, \sigma_i)$, and thus the action is transitive. □

The transitivity of the action allows us to identify each M_i as a permutation module induced by the stabilizer of our chosen base point. Let K_i denote the stabilizer of $(\mathbf{1}_{[k+1]}, \sigma_i)$ in $G^k \times G^{k+1}$.

Lemma 6.4. *For each $1 \leq i \leq k$, the stabilizer K_i of the element $(\mathbf{1}_{[k+1]}, \sigma_i)$ is given by*

$$K_i = \{(h, h * \sigma_i) \mid h \in G^k\}.$$

Proof. An element $(h, g) \in G^k \times G^{k+1}$ belongs to K_i if and only if $(h, g) \bullet (\mathbf{1}_{[k+1]}, \sigma_i) = (\mathbf{1}_{[k+1]}, \sigma_i)$. This condition is satisfied if and only if:

$$((h * \sigma_i) \cdot \mathbf{1}_{[k+1]} \cdot g^{-1}, \sigma_i) = (\mathbf{1}_{[k+1]}, \sigma_i).$$

Equating the first components, we obtain the equation $(h * \sigma_i) \cdot g^{-1} = \mathbf{1}_{[k+1]}$, which implies $g = h * \sigma_i$. □

In particular, the elements of K_i are in one-to-one correspondence with G^k. Note that if we write $h \in G^k$ as $h = (h_1, \dots, h_k)$, then

$$h * \sigma_i = (h_1, \dots, h_i, h_i, h_{i+1}, \dots, h_k)$$

.

Since M_i is a transitive permutation module, we have the following isomorphism of $(G^k \times G^{k+1})$-modules:

$$M_i \cong \operatorname{Ind}_{K_i}^{G^k \times G^{k+1}}(\mathrm{tr}_{K_i}).$$

For each $1 \leq i \leq k$, we define a group monomorphism $\varphi_i \colon G^k \to G^{k+1}$ by $\varphi_i(h) = h * \sigma_i$. Explicitly, for $h = (h_1, \dots, h_k) \in G^k$, we have

$$\varphi_i(h_1, \dots, h_k) = (h_1, \dots, h_i, h_i, h_{i+1}, \dots, h_k).$$

It is easy to verify that this is indeed a group monomorphism.

Proposition 6.5. *Let $U \in \mathrm{IRep}(G^k)$ and $V \in \mathrm{IRep}(G^{k+1})$. For each $1 \leq i \leq k$, the multiplicity of the simple $(G^k \times G^{k+1})$-module $U \boxtimes V^*$ in M_i is equal to the multiplicity of V in the induced module $Ind_{G^k}^{G^{k+1}}(U)$, where the induction is taken along the group homomorphism $\varphi_i \colon G^k \to G^{k+1}$. In terms of characters, this is expressed as:*

$$\langle U \boxtimes V^*, M_i \rangle_{G^k \times G^{k+1}} = \langle V, \operatorname{Ind}_{G^k}^{G^{k+1}} U \rangle_{G^{k+1}}.$$

Proof. Using the character of the permutation module M_i as identified in Lemma 6.4:

$$\langle U \boxtimes V^*, M_i \rangle_{G^k \times G^{k+1}} = \langle U \boxtimes V^*, \operatorname{Ind}_{K_i}^{G^k \times G^{k+1}} \mathrm{tr}_{K_i} \rangle_{G^k \times G^{k+1}}.$$

Applying Frobenius reciprocity:

$$\langle U \boxtimes V^*, \operatorname{Ind}_{K_i}^{G^k \times G^{k+1}} \mathrm{tr}_{K_i} \rangle_{G^k \times G^{k+1}} = \langle \operatorname{Res}_{K_i}^{G^k \times G^{k+1}} (U \boxtimes V^*), \mathrm{tr}_{K_i} \rangle_{K_i}.$$

As a summation, we have:

$$\begin{aligned}
\langle \operatorname{Res}_{K_i}^{G^k \times G^{k+1}} (U \boxtimes V^*), \operatorname{tr}_{K_i} \rangle_{K_i} &= \frac{1}{|G^k|} \sum_{h \in G^k} (U \boxtimes V^*)(h, h * \sigma_i) \cdot 1 \\
&= \frac{1}{|G^k|} \sum_{h \in G^k} (U(h) \cdot V^*(h * \sigma_i)) \\
&= \frac{1}{|G^k|} \sum_{h \in G^k} (U(h) \cdot \overline{V(h * \sigma_i)})
\end{aligned}$$

Note that $V(h * \sigma_i) = \operatorname{Res}_{G^k}^{G^{k+1}} V(h)$ where the restriction is taken along the homomorphism φ_i.

Therefore, the final expression becomes:

$$\frac{1}{|G^k|} \sum_{h \in G^k} U(h) \cdot \overline{\operatorname{Res}_{G^k}^{G^{k+1}} V(h)} = \langle U, \operatorname{Res}_{G^k}^{G^{k+1}} V \rangle_{G^k}.$$

Finally, applying Frobenius reciprocity again, we obtain:

$$\langle U, \operatorname{Res}_{G^k}^{G^{k+1}} V \rangle_{G^k} = \langle \operatorname{Ind}_{G^k}^{G^{k+1}} U, V \rangle_{G^{k+1}}.$$

This completes the proof. □

Clearly, since φ_i acts as the identity on all coordinates $j \neq i$, investigating the induction from G^k to G^{k+1} along φ_i reduces to understanding the induction from G to $G \times G$ along the diagonal homomorphism $d(g) = (g, g)$.

Lemma 6.6. *Let $U_1, U_2, U_3 \in \operatorname{IRep}(G)$. The multiplicity of $U_1 \boxtimes U_2$ in the induced module $\operatorname{Ind}_G^{G \times G} U_3$ is equal to the multiplicity of U_3 in the tensor product $U_1 \otimes U_2$ as a G-module.*

Proof. By Frobenius reciprocity for the group pair $(G, G \times G)$, we have:

$$\langle \operatorname{Ind}_G^{G \times G} U_3, U_1 \boxtimes U_2 \rangle_{G \times G} = \langle U_3, \operatorname{Res}_G^{G \times G} (U_1 \boxtimes U_2) \rangle_G.$$

The restriction of the external tensor product $U_1 \boxtimes U_2$ to the diagonal subgroup is the internal tensor product $U_1 \otimes U_2$. Substituting this into the inner product, we obtain:

$$\langle U_3, \operatorname{Res}_G^{G \times G} (U_1 \boxtimes U_2) \rangle_G = \langle U_3, U_1 \otimes U_2 \rangle_G.$$

□

Lemma 6.7. *Let $U = U_1 \boxtimes \cdots \boxtimes U_k \in IRep(G^k)$ and $V = V_1 \boxtimes \cdots \boxtimes V_{k+1} \in IRep(G^{k+1})$. The multiplicity of $U \boxtimes V^*$ in M_i is non-zero only if:*

- *$V_r \cong U_r$ for all $r < i$,*
- *$V_r \cong U_{r-1}$ for all $r > i + 1$.*

In this case, the multiplicity is given by the multiplicity of U_i in the tensor product $V_i \otimes V_{i+1}$.

Proof. From Proposition 6.5, we have:

$$\langle M_i, U \boxtimes V^* \rangle = \langle \operatorname{Ind}_{G^k}^{G^{k+1}} U, V \rangle_{G^{k+1}}.$$

Since the induction along φ_i is the identity on all components except the i-th one, where it is the diagonal induction, we have:

$$\operatorname{Ind}_{G^k}^{G^{k+1}} U \cong U_1 \boxtimes \cdots \boxtimes U_{i-1} \boxtimes \operatorname{Ind}_G^{G\times G}(U_i) \boxtimes U_{i+1} \boxtimes \cdots \boxtimes U_k.$$

Comparing this with $V = V_1 \boxtimes \cdots \boxtimes V_{k+1}$, the result follows from Lemma 6.6. □

If we use the Kronecker delta notation:

$$\delta_{U,V} = \begin{cases} 1 & \text{if } U \cong V \\ 0 & \text{otherwise} \end{cases}$$

then the multiplicity formula can be expressed compactly as:

$$\langle M_i, U \boxtimes V^* \rangle = \left(\prod_{r=1}^{i-1} \delta_{U_r, V_r} \right) \left(\prod_{r=i+1}^{k} \delta_{U_r, V_{r+1}} \right) \langle U_i, V_i \otimes V_{i+1} \rangle_G.$$

In order to obtain the number of arrows from V to U all is left is to sum this number over i from 1 to k. We can conclude:

Theorem 6.8. *The quiver Q of the algebra $\mathbb{C}(G \wr \mathrm{SEO}_n)$ or, equivalently, $\mathbb{C}(G \wr \mathrm{PO}_n)$ is constructed as follows:*

- ***Vertices:*** *The set of vertices is the disjoint union $\bigsqcup_{k=0}^{n} \mathrm{IRep}(G^k)$, where each vertex is a simple module of the form $U = U_1 \boxtimes \cdots \boxtimes U_k$ with $U_r \in \mathrm{IRep}(G)$.*
- ***Arrows:*** *For any $V = V_1 \boxtimes \cdots \boxtimes V_{k+1} \in \mathrm{IRep}(G^{k+1})$ and $U = U_1 \boxtimes \cdots \boxtimes U_k \in \mathrm{IRep}(G^k)$, the number of arrows from V to U is given by the sum:*

$$\sum_{i=1}^{k} \left(\prod_{r=1}^{i-1} \delta_{U_r, V_r} \right) \left(\prod_{r=i+1}^{k} \delta_{U_r, V_{r+1}} \right) \langle U_i, V_i \otimes V_{i+1} \rangle_G,$$

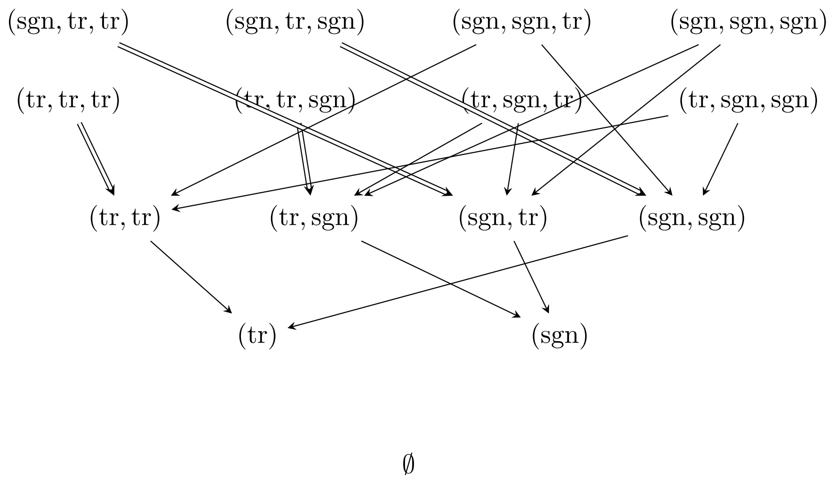
where $\langle U_i, V_i \otimes V_{i+1} \rangle_G$ is the multiplicity of U_i in the tensor product $V_i \otimes V_{i+1}$ as a G-module.

Example 6.9. We illustrate Theorem 6.8 by constructing the quiver Q of the algebra $\mathbb{C}(C_2 \wr \mathrm{PO}_3)$. The group $G = C_2$ has two simple modules: the trivial module tr and the sign module sgn.

The tensor product rules for these modules are straightforward:

⊗	tr	sgn
tr	tr	sgn
sgn	sgn	tr

The quiver is a bit crowded, and given by:



Note that a multiplicity of two between vertices is denoted by a double-lined arrow.

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