# A CRITICAL-SCALE EXTENSION OF ZHIZHIASHVILI'S THEOREM FOR RECTANGULAR FOURIER SERIES

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Abstract. We address a long-standing endpoint problem arising from Zhizhiashvili's logarithmic modulus theorem for multiple Fourier series. We prove an endpoint Dini criterion for almost-everywhere Pringsheim convergence of ordinary rectangular partial sums. In Zhizhiashvili's theorem the logarithmic modulus is assumed with exponent strictly above the critical value; here this strict power margin is replaced by a summable endpoint Dini condition. As a consequence, one obtains double-logarithmic endpoint classes lying outside the range of the classical theorem. The proof reduces the endpoint smoothness assumption to the Kaczmarz–Kojima product-logarithmic coefficient criterion by weighted translation-difference estimates.

## 1. Introduction

Almost-everywhere convergence of Fourier series has a fundamentally different nature in one and in several variables. In one dimension, the Carleson–Hunt theorem gives almost-everywhere convergence of the symmetric partial sums for every function in $L^p(\mathbb{T})$, $p > 1$ [1, 3]. In several variables, the geometry of the partial sums becomes part of the problem. The present note concerns ordinary symmetric rectangular sums and convergence in the sense of Pringsheim.

Throughout, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and $dx$ denotes the normalized Haar measure on $\mathbb{T}^d$; thus, on the fundamental cube,

$$dx = (2\pi)^{-d}\, dx_1 \cdots dx_d.$$

For $f \in L^1(\mathbb{T}^d)$ put

$$\widehat{f}(k) = \int_{\mathbb{T}^d} f(x) e^{-ik\cdot x}\, dx, \qquad k = (k_1, \ldots, k_d) \in \mathbb{Z}^d.$$

The ordinary rectangular partial sums are

$$S_{\mathbf{n}} f(x) = \sum_{|k_1| \le n_1} \cdots \sum_{|k_d| \le n_d} \widehat{f}(k) e^{ik\cdot x}, \qquad \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d. \tag{1.1}$$

2020 *Mathematics Subject Classification.* Primary 42B05; Secondary 42A20, 42B08.

*Key words and phrases.* Multiple Fourier series, rectangular partial sums, Pringsheim convergence, almost-everywhere convergence, logarithmic modulus of continuity, Zhizhiashvili theorem, Kaczmarz–Kojima theorem.

We say that $S_{\mathbf{n}} f$ converges to $f$ in the Pringsheim sense if

$$S_{\mathbf{n}} f(x) \longrightarrow f(x) \qquad \text{as } \min_{1\le j\le d} n_j \to \infty.$$

This is a genuinely multi-parameter question. Fefferman's divergence theorem shows that the higher-dimensional theory contains divergence phenomena absent from the one-dimensional theory [2]. It is therefore natural to ask which quantitative smoothness assumptions recover almost-everywhere convergence at the borderline of the logarithmic scale.

For $h = (h_1, \dots, h_d) \in \mathbb{T}^d$, we write

$$|h| = (h_1^2 + \cdots + h_d^2)^{1/2},$$

where $h_j \in [-\pi, \pi)$ is the representative near the origin. For $f \in L^2(\mathbb{T}^d)$ let

$$\omega_2(f, \delta) = \sup_{|h|\le\delta} \|f(\cdot + h) - f(\cdot)\|_{L^2(\mathbb{T}^d)}, \qquad 0 < \delta < 1. \tag{1.2}$$

Zhizhiashvili proved the following fundamental logarithmic criterion: if $d \ge 2$, $f \in L^2(\mathbb{T}^d)$, and

$$\omega_2(f, \delta) = O\left(\left(\log \frac{1}{\delta}\right)^{-\beta}\right) \qquad (\delta \downarrow 0) \tag{1.3}$$

for some $\beta > d/2$, then the rectangular Fourier sums of $f$ converge almost everywhere in the Pringsheim sense [6]. The strict inequality $\beta > d/2$ is the decisive feature of this theorem. It leaves open the critical logarithmic exponent itself.

The main result of this paper gives a partial endpoint solution to this problem by replacing the missing logarithmic power margin with a Dini summability condition.

**Theorem 1.1** (Endpoint Zhizhiashvili–Dini criterion)**.** *Let $d \ge 2$ and $f \in L^2(\mathbb{T}^d)$. Suppose that, for some $\eta \in (0, 1)$,*

$$\int_0^\eta \omega_2(f, t)^2 \left(\log \frac{e}{t}\right)^{d-1} \frac{dt}{t} < \infty. \tag{1.4}$$

*Then*

$$S_{\mathbf{n}} f(x) \longrightarrow f(x) \qquad \textit{for almost every } x \in \mathbb{T}^d$$

*in the Pringsheim sense.*

Theorem 1.1 is a significant endpoint advance of Zhizhiashvili's theorem in the $L^2$ scale. The classical hypothesis (1.3) with $\beta > d/2$ immediately implies (1.4). More importantly, Theorem 1.1 also includes moduli lying exactly at the critical power $d/2$, provided that the remaining endpoint contribution is summable. The following consequence records the simplest explicit form of this gain.

**Corollary 1.2** (Double-logarithmic endpoint class). *Let $d \geq 2$, $f \in L^2(\mathbb{T}^d)$, and $\gamma > 0$. If*

$$\omega_2(f,\delta) = O\left(\left(\log \frac{e}{\delta}\right)^{-d/2} \left(\log\log \frac{e^e}{\delta}\right)^{-1/2-\gamma}\right) \qquad (\delta \downarrow 0), \tag{1.5}$$

*then $S_{\mathbf{n}} f(x) \to f(x)$ for almost every $x \in \mathbb{T}^d$ in the Pringsheim sense.*

Corollary 1.2 is not a restatement of Zhizhiashvili's result. It permits the critical logarithmic power $d/2$ and compensates only by a summable iterated logarithm. Thus it applies to moduli which may decay more slowly than every power

$$\left(\log \frac{1}{\delta}\right)^{-d/2-\varepsilon}, \qquad \varepsilon > 0,$$

and hence are outside the range of (1.3).

The proof is short and structural. We use the Kaczmarz–Kojima theorem, which says that a product-logarithmic square summability condition on the Fourier coefficients implies almost-everywhere rectangular convergence. The new point is that the endpoint Dini condition (1.4) forces exactly this coefficient condition. A weighted lower bound for the oscillatory multipliers $e^{imt} - 1$ converts one-coordinate translation estimates into one-coordinate logarithmic coefficient estimates. The arithmetic–geometric mean inequality then converts the resulting family of estimates into the Kaczmarz–Kojima product weight.

## 2. Preliminaries

All logarithms are natural. Constants denoted by $C$ may change from line to line and may depend on fixed parameters such as $d$ and $\eta$, but not on the frequency variable. We use Plancherel's theorem and uniqueness of trigonometric Fourier coefficients in their standard forms; see, for instance, [7].

For a coordinate vector $e_j$, define

$$\Delta_j(t) f(x) = f(x + te_j) - f(x), \qquad 1 \leq j \leq d.$$

We shall use the following classical convergence theorem.

**Theorem 2.1** (Kaczmarz–Kojima product-log theorem). *Let $d \geq 2$ and $F \in L^2(\mathbb{T}^d)$. If*

$$\sum_{k \in \mathbb{Z}^d} \left|\widehat{F}(k)\right|^2 \prod_{j=1}^{d} \log(|k_j| + 2) < \infty, \tag{2.1}$$

*then the rectangular partial sums $S_{\mathbf{n}} F$ converge to $F$ almost everywhere in the Pringsheim sense.*

For $d = 2$ this is due to Kaczmarz; the higher-dimensional form is due to Kojima [4, 5].

## 3. Weighted differences and logarithmic coefficients

The next elementary estimate extracts a full logarithmic power from a one-dimensional translation difference.

**Lemma 3.1.** *Let $d \geq 1$ and $0 < \eta < 1$. There exists $c = c(d,\eta) > 0$ such that, for every integer $m$ with $|m| \geq 1$,*

$$\int_0^\eta \left|e^{imt} - 1\right|^2 \left(\log \frac{e}{t}\right)^{d-1} \frac{dt}{t} \geq c \log^d(|m|+2). \tag{3.1}$$

*Proof.* By symmetry it is enough to consider $m = N \geq 1$. For every finite range $1 \leq N \leq N_0(d,\eta)$ the assertion is absorbed by decreasing the constant, since the left-hand side is positive for each fixed $N \geq 1$. Hence we assume that $N$ is large.

Choose $0 < \alpha < \pi/8$. For integers $q \geq 0$ set

$$I_q = \{t > 0 : |Nt - (2q+1)\pi| < \alpha\}.$$

On $I_q$ we have $\left|e^{iNt} - 1\right|^2 \geq c_0$. If $0 \leq q \leq Q$, where $Q = \lfloor c_1 N\eta \rfloor$ and $c_1 > 0$ is chosen sufficiently small, then $I_q \subset (0,\eta)$ for all large $N$. Moreover, on $I_q$ one has $t \asymp (q+1)/N$ and $|I_q| \asymp N^{-1}$. Hence

$$\begin{aligned} \int_0^\eta \left|e^{iNt} - 1\right|^2 \left(\log \frac{e}{t}\right)^{d-1} \frac{dt}{t} &\geq C \sum_{q=0}^{Q} \frac{1}{q+1} \left(\log \frac{eN}{C(q+1)}\right)^{d-1} \\ &\geq C \int_1^{c_2 N} \left(\log \frac{eN}{Cu}\right)^{d-1} \frac{du}{u}. \end{aligned}$$

The change of variables $v = \log(eN/(Cu))$ shows that the last integral is bounded from below by $C(\log N)^d - C_{d,\eta}$. This is at least $c(d,\eta)\log^d(N+2)$ for all sufficiently large $N$, and the remaining finite range has already been absorbed into the constant. ☐

The next proposition is the main reduction step. It converts the endpoint Dini control of coordinate translation differences into the product-logarithmic Fourier coefficient condition needed for the Kaczmarz–Kojima convergence theorem.

**Proposition 3.2.** *Let $d \geq 2$ and $f \in L^2(\mathbb{T}^d)$. Suppose that, for some $\eta \in (0,1)$ and every coordinate $j = 1,\ldots,d$,*

$$\int_0^\eta \|\Delta_j(t)f\|_2^2 \left(\log \frac{e}{t}\right)^{d-1} \frac{dt}{t} < \infty. \tag{3.2}$$

*Then*

$$\sum_{k\in\mathbb{Z}^d} \left|\widehat{f}(k)\right|^2 \prod_{j=1}^{d} \log(|k_j|+2) < \infty. \tag{3.3}$$

*Proof.* Fix a coordinate $j$. Since $f \in L^2(\mathbb{T}^d)$,

$$\widehat{\Delta_j(t)f}(k) = (e^{ik_j t} - 1)\widehat{f}(k), \qquad k \in \mathbb{Z}^d,$$

and Plancherel's theorem gives, for every $t$,

$$\|\Delta_j(t)f\|_2^2 = \sum_{k\in\mathbb{Z}^d} \left|\widehat{f}(k)\right|^2 \left|e^{ik_jt} - 1\right|^2.$$

Multiplying by $(\log(e/t))^{d-1}/t$ and applying Tonelli's theorem, the assumption (3.2) implies

$$\sum_{k\in\mathbb{Z}^d} \left|\widehat{f}(k)\right|^2 \int_0^\eta \left|e^{ik_jt} - 1\right|^2 \left(\log\frac{e}{t}\right)^{d-1} \frac{dt}{t} < \infty. \tag{3.4}$$

Lemma 3.1 yields

$$\sum_{\substack{k\in\mathbb{Z}^d\\|k_j|\geq 1}} \left|\widehat{f}(k)\right|^2 \log^d(|k_j|+2) < \infty. \tag{3.5}$$

Because $f \in L^2(\mathbb{T}^d)$, Plancherel gives $\sum_k \left|\widehat{f}(k)\right|^2 < \infty$. Therefore the terms with $k_j = 0$ may be added to (3.5), and for every $j$,

$$\sum_{k\in\mathbb{Z}^d} \left|\widehat{f}(k)\right|^2 \log^d(|k_j|+2) < \infty. \tag{3.6}$$

For non-negative $a_1, \ldots, a_d$,

$$\prod_{j=1}^d a_j \leq \frac{1}{d}\sum_{j=1}^d a_j^d.$$

Taking $a_j = \log(|k_j|+2)$, multiplying by $\left|\widehat{f}(k)\right|^2$, and summing over $k$ gives (3.3) by (3.6). □

## 4. Proofs of the endpoint results

*Proof of Theorem 1.1.* For each coordinate $j$ and every $t > 0$,

$$\|\Delta_j(t)f\|_2 \leq \omega_2(f,t).$$

Thus the endpoint Dini assumption (1.4) implies (3.2) for every coordinate. Proposition 3.2 gives the Kaczmarz–Kojima product-log condition (3.3). The convergence conclusion follows from Theorem 2.1. □

*Proof of Corollary 1.2.* For small $t$, the hypothesis (1.5) gives

$$\omega_2(f,t)^2 \left(\log\frac{e}{t}\right)^{d-1} \frac{1}{t} \leq C\frac{1}{t\log(e/t)\left(\log\log(e^e/t)\right)^{1+2\gamma}}.$$

With $u = \log(e/t)$, the right-hand side is integrable near the origin, since the corresponding integral is bounded by a constant multiple of

$$\int^\infty \frac{du}{u(\log u)^{1+2\gamma}} < \infty.$$

Hence (1.4) holds, and Theorem 1.1 applies. □

**Corollary 4.1** (Recovery of Zhizhiashvili's $L^2$ theorem)**.** *Let $d \geq 2$ and $f \in L^2(\mathbb{T}^d)$. If (1.3) holds for some $\beta > d/2$, then $S_{\mathbf{n}}f(x) \to f(x)$ almost everywhere in the Pringsheim sense.*

*Proof.* After the change of variables $u = \log(e/t)$, the integral in (1.4) is dominated by a constant multiple of

$$\int^{\infty} u^{d-1-2\beta}\, du,$$

which is finite precisely when $\beta > d/2$. The assertion follows from Theorem 1.1. □

*Remark* 4.2. Corollary 4.1 shows that the classical theorem of Zhizhiashvili is contained in the endpoint Dini criterion. Corollary 1.2 shows the strict endpoint gain: at the critical logarithmic exponent, a summable iterated-logarithmic improvement is sufficient. This is the precise sense in which Theorem 1.1 strengthens Zhizhiashvili's logarithmic modulus criterion.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Data Availability

Not applicable.

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