TWISTED JACQUET MODULES ASSOCIATED TO MAXIMAL PARABOLIC SUBGROUPS AND CUSPIDAL REPRESENTATIONS OF $\mathrm{GL}(n, q)$

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Abstract. Let π be a cuspidal representation $\mathrm{GL}(n, F)$ over a finite field F. Let $P = MN$ be the Levi decomposition of a maximal parabolic subgroup corresponding to the partition $(k, n-k)$ of n. Given a rank r character ψ_r of the unipotent radical N, the twisted Jacquet module π_{N,ψ_r} is a representation of the subgroup M_r of M which stabilizes ψ_r. The main problem we solve in this work is to determine the structure of π_{N,ψ_r} as a M_r-module. This problem was first studied by D. Prasad, who solved the problem for the case $r = k = n/2$. This and subsequent works on the problem for special cases of (r, k, n), identify the structure of π_{N,ψ_r} by calculating its character and matching it to a known representation of M_r. In this work we solve the problem for all values of (r, k, n) directly without calculating the character of π_{N,ψ_r}. Our solution depends on two other key conceptual advances: (i) We show that the twisted Jacquet functor which takes a complex representation of P to its twisted Jacquet modules (one for each rank), gives an equivalence of categories between $\mathrm{Rep}(P)$ and the direct sum $\oplus_r \mathrm{Rep}(M_r)$ of the categories $\mathrm{Rep}(M_r)$. (ii) We use this equivalence to construct a recursively defined representation $\Pi_{k,n}$ of P, which generalizes to P, the representation of the Mirabolic subgroup obtained from the trivial representation by iterating the Bernstein-Zelevinsky Φ^+ functor. Like the representation $(\Phi^+)^{n-1}(1)$ of the Mirabolic subgroup, the representation $\Pi_{n-k,n}$ (after composing with the inverse transpose isomorphism) satisfies a universal property with respect to restrictions to P of cuspidal representations of G_n. Our solution of the main problem is a simple consequence of this universal property.

1. Introduction

Let F be the finite field with q elements and let $\mathrm{GL}(n, F)$ be the general linear group. Let P be a parabolic subgroup of $\mathrm{GL}(n, F)$ with Levi decomposition $P = MN$, where N is the unipotent radical of P. Let (π, V) be an irreducible complex representation of $\mathrm{GL}(n, F)$ and ψ be a character of N. For $m \in M$ and a character ψ of N, let ψ^m be the character of N defined by $\psi^m(n) = \psi(m^{-1}nm)$. Let

$$V_{N,\psi} = \{v \in V : \pi(n)v = \psi(n)v, \forall n \in N\},$$

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be the ψ-isotypic component of the restriction of π to N. It is easy to see that $V_{N,\psi}$ is invariant under the subgroup M_ψ of M given by

$$M_\psi = \{m \in M \colon \psi^m(n) = \psi(n), \forall n \in N\}.$$

The representation $(\pi_{N,\psi}, V_{N,\psi})$ of M_ψ is known as the *twisted Jacquet module* of π with respect to (N, ψ). It is an interesting question to determine for which irreducible representations π of $\mathrm{GL}(n, F)$ and characters ψ of N, the twisted Jacquet module $\pi_{N,\psi}$ is non-zero and to understand its structure as a representation of M_ψ.

Before we continue, we set up some more notation and state the problem we study in this paper. We denote $\mathrm{GL}(n, F)$ as G_n. Let π be a cuspidal representation of G_n, and let P be a maximal parabolic subgroup of G_n corresponding to the partition $(k, n-k)$ of n. We have $M \cong G_k \times G_{n-k}$ and $N \cong \mathrm{M}_{k\times n-k}(F)$. It is easy to see that the characters of N are of the form $\psi_A(X) = \psi_0(\mathrm{Tr}(A^\top X))$, where $\psi_0 : F \to \mathbb{C}^\times$ is a non-trivial additive character of F and $A \in \mathrm{M}_{k\times n-k}(F)$. The character ψ_A is said to be of rank r if the matrix A has rank r. It is said to be non-degenerate if A has full rank $\min\{k, n-k\}$, and degenerate otherwise.

The main result of this paper (see Theorem 5.1) is the solution of the following problem:

Problem $(\mathrm{Prob}_{k,n,r,\pi})$**.** *Let π be an irreducible cuspidal representation of G_n, and let P be a maximal parabolic subgroup of G_n associated with the partition $(k, n-k)$ of n. Let $P = MN$ be the Levi decomposition of P, and let ψ be a rank r character of N. Determine the structure of the twisted Jacquet module $\pi_{N,\psi}$ as a M_ψ-module.*

The study of this problem was initiated by D. Prasad in [Pra00], and it is the main inspiration for this work. In [Pra00], $\mathrm{Prob}_{k,2k,k,\pi}$ was solved for the case $n = 2k$ and a non-degenerate character ψ of rank k, by first computing the character of $\pi_{N,\psi}$, and showing that it equals the character of a known representation of M_ψ. The subsequent works [BK22a], [BK23], [BK24],[BDK25], solve the problem $\mathrm{Prob}_{k,n,r,\pi}$ for $r \in \{1, 2\}$ and some small values of k, n. The above problem has a natural generalization $\mathrm{Prob}_{\lambda\vdash n,n,r,\pi}$ to the case when we replace the maximal parabolic subgroup P of G_n with a standard parabolic subgroup for an arbitrary partition $\lambda \vdash n$ of n, and for an appropriate notion of rank of a character of N. The special case when the parts of λ have equal size, and the character of N is non-degenerate was solved by O. Gorodetsky and Z. Hazan in [GH19], again by first carrying out the elaborate calculation of the character of $\pi_{N,\psi}$, and then matching it to the character of a known representation of M_ψ. In another direction, an analogue of the problem $\mathrm{Prob}_{2,4,2,\pi}$ with the finite field F replaced by a finite principal ideal local ring of length 2 was studied in [PP25]. The structure of the twisted Jacquet module for representations of many finite classical groups is still unknown. In a recent work [Coh26], the dimension of the module for any irreducible representation of $\mathrm{GSp}(4, q)$ with respect to a non-degenerate character of the unipotent radical of the Siegel parabolic subgroup was determined. However, not much is known about the structure of the module for cuspidal representations even for $\mathrm{GL}(n, F)$ where F is a p-adic field.

In our solution of the general problem $\mathrm{Prob}_{k,n,r,\pi}$ (see Theorem 5.1), we obtain the structure of $\pi_{N,\psi}$ directly, bypassing the need to carry out the elaborate calculation of the character of $\pi_{N,\psi}$. In order to explain the key idea of proof, let us assume here the condition that $k < n$ (which can be relaxed). Assuming this, we construct a fundamental representation $\Pi_{k,n}$ of $P_{k,n}$ which has the property that for any cuspidal representation $\pi = \pi_\theta$ of G_n, there is a cuspidal representation $\pi_{\tilde{\theta}}$ of G_k such that $\pi_\theta|_{P_{k,n}} = \pi_{\tilde{\theta}} \otimes \Pi_{k,n}$, where we think of $P_{k,n} = \left(\begin{smallmatrix} G_k & * \\ & G_{n-k} \end{smallmatrix}\right)$, and by $\pi_{\tilde{\theta}}$, we mean its inflation from G_k to $P_{k,n}$. We show that the problem reduces to determining the twisted Jacquet module of $\Pi_{k,n}$. We are immediately able to determine $\pi_{N,\psi}$ because, the very construction of $\Pi_{k,n}$ (see Definition 4.3) is by prescribing its twisted Jacquet modules in a way that ensures the desired property $\pi_\theta|_{P_{k,n}} = \pi_{\tilde{\theta}} \otimes \Pi_{k,n}$, see Theorem 4.9 and part (1) of Theorem 4.13): this in turn depends on our Theorem 4.1, which proves that a representation of $P_{k,n}$ can be synthesized from its twisted Jacquet modules (of all ranks). More generally, we show in Section 4.1, that the functor $\mathfrak{F} : \mathrm{Rep}(P_{k,n}) \to \oplus_{r=0}^{k} \mathrm{Rep}(M_r)$ that takes a representation of $P_{k,n}$ to its twisted Jacquet modules (one for each rank), is an equivalence of categories. The recursive definition of the representation $\Pi_{k,n}$ should be seen as generalizing the representation of the Mirabolic group Mir_n, obtained from the trivial representation by iterating (recursively) the Bernstein-Zelevinsky Φ^+-functor.

In the case when F is a p-adic field, our work raises the following natural questions which we pose to the community:

1. Is there an analogous construction of the representation $\Pi_{k,n}$ of Definition 4.3?
2. If so, is there a generalization of the universal property in part (1) of Theorem 4.13, for the distribution character of a cuspidal representation of $\mathrm{GL}(n, F)$?

We hope that understanding these questions can shed light on the structure of the module $\pi_{N,\psi}$ for p-adic groups.

This paper is organized as follows. In §2, we set up the notation and record some preliminary representation theoretic results needed for the sequel. In §3, we record some combinatorial results that we will need. In §4.1, we show (see Theorem 4.1) that the twisted Jacquet functor gives an equivalence between the categories $\mathrm{Rep}(P)$ and $\oplus_r \mathrm{Rep}(M_r)$ where $M_r = M_{\psi_r}$ for a rank r-character ψ_r of N. We then construct in §4.2, a representation $\Pi_{k,n}$ of $P = P_{k,n}$ (see Definition 4.3) that satisfies a universal property with respect to restriction of cuspidal representations of G_n to $P_{k,n}$ (see Theorem 4.13). In §5, we use the representation $\Pi_{k,n}$ to obtain our solution of $\mathrm{Prob}_{k,n,r,\pi}$ (see Theorem 5.1).

2. Preliminaries

We begin with a list of some symbols and notations that we frequently use.

F: A finite field of order q. The symbol F_m denotes the finite field of order q^m. The symbol $F_m^\times$ denotes the multiplicative group of F_m.

$\phi(x) = x^q$: is the Frobenius automorphism of F_m. The Galois group $\mathrm{Gal}(F_m/F)$ is the cyclic group of order m generated by ϕ.

$N_{F_m/F}(x) = x\phi(x)\cdots\phi^{m-1}(x)$**:** is the norm of $x \in F_m$ over F.

$\mathrm{Tr}_{F_m/F}(x) = x + \phi(x) + \cdots + \phi^{m-1}(x)$**:** is the trace of $x \in F_m$ over F.

$\widehat{\mathcal{G}}$**:** refers to the set of isomorphism classes of irreducible representations of a finite group $\mathcal{G}$. By a representation, we always mean a finite dimensional complex representation.

$\mathrm{Rep}(\mathcal{G})$**:** denotes the category whose objects are finite dimensional $\mathbb{C}$-representations of the finite group $\mathcal{G}$, and whose morphisms are the intertwining operators between the representations.

χ_ρ**:** denotes the character of a representation ρ of a finite group $\mathcal{G}$. We implicitly use the fact that two representations of $\mathcal{G}$ are isomorphic if and only if their characters are equal.

$\langle \chi, \chi' \rangle_{\mathcal{G}}$**:** denotes the inner product $(\sum_{g\in\mathcal{G}} \chi(g)\overline{\chi'(g)})/|\mathcal{G}|$ of characters χ, χ' of $\mathcal{G}$.

$\rho|_H$**:** is the restriction of a representation ρ of a group $\mathcal{G}$ to its subgroup H. More generally, given a subset $Y \subset X$ and a function f on X, we denote the restriction of f to Y by $f|_Y$.

$\mathrm{Ind}_H^{\mathcal{G}}$**:** denotes the induced representation/character. We implicitly use Frobenius reciprocity: $\langle \chi', \mathrm{Ind}_H^{\mathcal{G}} \chi \rangle_{\mathcal{G}} = \langle \chi'|_H, \chi \rangle_H$.

ψ_0**:** a fixed non-trivial additive character of F.

$\mathrm{M}_{k\times\ell}(F)$**:** the additive group (or F-vector space) of $k \times \ell$ matrices over F.

$\mathrm{Tr}(X)$**:** the trace of a square matrix X.

$X^\top$**:** the transpose of a matrix X.

G_n**:** is the general linear group $\mathrm{GL}(n, F)$.

I_n**:** is the identity matrix in G_n.

$P_{r,n}$ **=:** $\{\left(\begin{smallmatrix} g_1 & g_3 \\ & g_2 \end{smallmatrix}\right) : g_1 \in G_r, g_2 \in G_{n-r}, g_3 \in \mathrm{M}_{r\times n-r}(F)\}$ is the parabolic subgroup of G_n for the partition $(r, n-r)$ of n.

$P_{r,n}^*$ **=:** $\{\left(\begin{smallmatrix} g_1 & \\ g_3 & g_2 \end{smallmatrix}\right) : g_1 \in G_r, g_2 \in G_{n-r}, g_3 \in \mathrm{M}_{n-r\times r}(F)\}$ is the image of $P_{r,n}$ under the inverse transpose isomorphism of G_n.

2.1. Regular characters of $F_n^\times$ and cuspidal representations of G_n.

Let γ be a generator of the cyclic group $F_n^\times$, and let $\widehat{F_n^\times}$ be the group of characters of $F_n^\times$. Let $\imath_n \colon F_n^\times \to \widehat{F_n^\times}$ be the group isomorphism

$$\imath_n(\gamma^j) = \theta_j, \qquad \theta_j(\gamma^s) = \exp(\tfrac{2\pi\sqrt{-1}js}{q^n-1}).$$

Let $\mathcal{F}_n \subset F_n$ be the elements of $F_n^\times$ which are not contained in any proper subfield of F_n. Alternatively, $\mathcal{F}_n \subset F_n^\times$ consists of elements whose $\mathrm{Gal}(F_n/F)$-orbit has size n, where the Galois group $\mathrm{Gal}(F_n/F)$ is the cyclic group of order n generated by the automorphism $\phi(x) = x^q$. The *regular characters* of $F_n^\times$, denoted $(\widehat{F_n^\times})_{\mathrm{reg}}$, consist of $\imath_n(\mathcal{F}_n)$. If F_d is a subfield of F_n, then $\beta = N_{F_n/F_d}(\gamma) = \gamma^{(q^n-1)/(q^d-1)}$ is a generator of the cyclic group $F_d^\times$. Again, we have an isomorphism

$$F_d^\times \xrightarrow{\imath_d} \widehat{F_d^\times}, \quad \imath_d(\beta^j) = \varphi_j, \quad \varphi_j(\beta^s) = \exp(\tfrac{2\pi\sqrt{-1}js}{q^d-1}).$$

Lemma 2.1.

(1) *Any character of* $F_d^\times$ *can be obtained by restricting a character of* $F_n^\times$ *to* $F_d^\times$. *Moreover, we have a commutative square:*

$$\begin{tikzcd}
F_n^\times \arrow[r, equal, "\imath_n"] \arrow[d, two heads, "N_{F_n/F_d}"'] & \widehat{F_n^\times} \arrow[d, two heads, "\mathrm{Res}^{F_n^\times}_{F_d^\times}"] \\
F_d^\times \arrow[r, equal, "\imath_d"'] & \widehat{F_d^\times}
\end{tikzcd}$$

(2) *If* d *is a proper divisor of* n*, then any character of* $F_d^\times$ *can be obtained by restricting a* <u>*regular*</u> *character of* $F_n^\times$ *to* $F_d^\times$*:*

$$\begin{tikzcd}
\mathcal{F}_n \arrow[r, equal, "\imath_n"] \arrow[d, two heads, "N_{F_n/F_d}"'] & (\widehat{F_n^\times})_{\mathrm{reg}} \arrow[d, two heads, "\mathrm{Res}^{F_n^\times}_{F_d^\times}"] \\
F_d^\times \arrow[r, equal, "\imath_d"'] & \widehat{F_d^\times}
\end{tikzcd}$$

Proof. (1) The element $\varphi_j = (\imath_d \circ N_{F_n/F_d})(\gamma^j)$ of $\widehat{F_d^\times}$ as defined above, is clearly the restriction of θ_j to $F_d^\times$, i.e., $(\mathrm{Res}^{F_n^\times}_{F_d^\times} \circ \imath_n)(\gamma^j)$.

(2) Here d is a proper divisor of n. In order to show that any character of $F_d^\times$ can be obtained by restricting a *regular* character of $F_n^\times$ to $F_d^\times$, it suffices to show the map $N_{F_n/F_d} : \mathcal{F}_n \to F_d^\times$ is surjective. Let e be a proper divisor of n such that $d|e$. It suffices to show $N_{F_n/F_e}(\mathcal{F}_n) = F_e^\times$, because $N_{F_e/F_d}(F_e^\times) = F_d^\times$. Choosing a maximal such e, and replacing d with e, we may now assume that $\dim_{F_d}(F_n)$ is a prime number ℓ. In this case $\mathcal{F}_n = F_n \setminus F_d$. For each $c \in F_d^\times$ there are $1 + q + \cdots + q^{\ell-1}$ solutions in F_n to the equation $N_{F_n/F_d}(x) = c$ of which there are at most ℓ in F_d, therefore there are $(q-1) + (q^2-1) + \cdots + (q^{\ell-1}-1) \geq q-1 > 0$ solutions in $\mathcal{F}_n$. ☐

We recall that the cuspidal representations of G_n arising from regular characters $\theta = \imath_n(x), \theta' = \imath_n(x')$ of $F_n^\times$ are isomorphic if and only if $x, x' \in \mathcal{F}_n$ are in the same $\mathrm{Gal}(F_n/F)$ orbit. The $\mathrm{Gal}(F_n/F)$-orbits on $\mathcal{F}_n$ correspond (via the minimal polynomial) to the set of roots of monic irreducible polynomials of degree n in $F[X]$. It is well known that the number of such polynomials (and hence, also the number of isomorphism classes of cuspidal representations of G_n) is $(-\delta_{1,n} + \sum_{d|n} \mu(d) q^{n/d})/n$, where $\mu(n)$ is the Möbius function, and δ_{ij} is the Kronecker delta.

For $a \in F_n$, let m_a be the F-linear endomorphism of F_n given by $x \mapsto ax$. Clearly $a \mapsto m_a$ is a monomorphism from $F_n^\times \to G_n$ (once a basis for F_n/F is chosen). In this way, we view $F_n^\times$ as a subgroup of G_n. An element $g \in G_n$ is said to *come from* $F_n^\times$ if there exists $a \in F_n$ and a basis of F_n/F such that g is the matrix of m_a with respect to this basis. Equivalently, $g \in G_n$ comes from $F_n^\times$

if and only if its minimal polynomial $p(X) \in F[X]$ is an irreducible polynomial of degree $m|n$.

Definition 2.2. For a character θ of $F_n^\times$, let Θ_θ be the class function on G_n defined as follows. Let $g \in G_n$ with $g = s \cdot u$ being the Jordan decomposition of g into its semisimple part s and unipotent part u. In case s comes from $F_n^\times$, let $\lambda \in F_m$ be a root of the irreducible minimal polynomial $p(X) \in F[X]$ of s, and let $t(g) = \dim_{F_n} \ker(g - \lambda I_n)$. We define:

$$\Theta_\theta(g) = \begin{cases} 0 & \text{if } s \text{ does not come from } F_n^\times \\ (-1)^{n-1} \left(\sum_{i=0}^{m-1} \theta(\phi^i(\lambda))\right) \prod_{j=1}^{t(g)-1}(1 - q^{jm}) & \text{if } s \text{ comes from } F_n^\times. \end{cases} \tag{2.1}$$

We recall the character of the cuspidal representation π_θ of G_n associated with the regular character θ of $F_n^\times$ (see [Gel70, Pra00]):

Theorem 2.1 (J.A. Green)**.** *Let θ be a regular character of $F_n^\times$. Let $\pi = \pi_\theta$ be the cuspidal representation of G_n associated to θ. The character of π_θ equals the class function Θ_θ given in Definition 2.2 above.*

2.2. Characters of N and the associated twisted Jacquet modules.

We elaborate some definitions from Section 1. Let $P = P_{k,n}$ be the parabolic subgroup of G_n with respect to the partition $(k, n-k)$ of n:

$$P = P_{k,n} = \{\left(\begin{smallmatrix} g_1 & Y \\ & g_2 \end{smallmatrix}\right) : g_1 \in G_k,\, g_2 \in G_{n-k},\, Y \in \mathrm{M}_{k \times n-k}(F)\}.$$

Let $P = MN$ be the Levi decomposition of P, where the unipotent radical N and the Levi subgroup M are:

$$N = \{g \in P \colon g_1 = I_k, g_2 = I_{n-k}\}, \quad M = \{g \in P \colon Y = 0\} \simeq G_k \times G_{n-k}.$$

Since $N \simeq \mathrm{M}_{k \times n-k}(F)$, the group $\widehat{N}$ of characters of N is

$$\widehat{N} = \{\psi_A \colon A \in \mathrm{M}_{k \times n-k}(F)\}, \quad \text{where } \psi_A(g) = \psi_0(\mathrm{Tr}(A^\top Y)).$$

We define the rank of the character ψ_A to be the rank of A. We say ψ_A is non-degenerate if A has full rank, and we say ψ_A is a degenerate character if A does not have full rank.

Let (τ, V) be a representation of $P \subset G_n$, and let ψ be a character of N. For $m \in M$, let ψ^m be the character of N defined by $\psi^m(n) = \psi(m^{-1}nm)$. Let

$$V_{N,\psi} = \{v \in V \colon \tau(n)v = \psi(n)v, \forall n \in N\},$$

be the ψ-isotypic component of $\tau|_N$. It is easy to see that the linear map $\mathrm{Pr}_N : V \to V$ defined by

$$\mathrm{Pr}_N = \frac{1}{|N|} \sum_{n \in N} \tau(n)\overline{\psi(n)},$$

is a projection of V onto $V_{N,\psi}$. Therefore, $V = V_{N,\psi} \oplus V(N,\psi)$ where $V(N,\psi) = \ker(\mathrm{Pr}_N) = \mathrm{Span}_{\mathbb{C}}\{\tau(n)v - \psi(n)v \colon n \in N, v \in V\}$. Clearly, $\tau(m)(V_{N,\psi}) = V_{N,\psi}$ for m in the subgroup M_ψ of M defined by:

$$M_\psi = \{m \in M \colon \psi^m(n) = \psi(n), \forall n \in N\}.$$

The map $\tau_{N,\psi} : M_\psi \to \mathrm{GL}(V_{N,\psi})$ defined by

$$\tau_{N,\psi}(m)(v) = \tau(m)v,\, m \in M_\psi$$

yields a representation of M_ψ, called the *twisted Jacquet module* of τ with respect to (N,ψ), and is denoted by $(\tau_{N,\psi}, V_{N,\psi})$. Let $\chi_{\tau_{N,\psi}}$ denote the character of $\tau_{N,\psi}$. The character $\chi_{\tau_{N,\psi}}$ can be calculated in terms of the character χ_τ by the following well known result (see for example [BK22b]). We include a proof here, for the sake of completeness:

Proposition 2.3. *Let (τ, V) be a representation of a parabolic subgroup $P \subset G_n$, and let χ_τ be the character of τ. For $m \in M_\psi$, we have*

$$\chi_{\tau_{N,\psi}}(m) = \frac{1}{|N|}\sum_{n\in N}\chi_\tau(mn)\overline{\psi(n)}.$$

Proof. The character $\chi_{\tau_{N,\psi}}(m)$ is the trace of the restriction of $\tau(m)$ to $V_{N,\psi}$, which is also equal to the trace of $\tau(m)\mathrm{Pr}_N$. The latter quantity equals

$$\mathrm{Tr}\left(\tfrac{1}{|N|}\sum_{n\in N}\tau(m)\tau(n)\overline{\psi(n)}\right) = \tfrac{1}{|N|}\sum_{n\in N}\chi_\tau(mn)\overline{\psi(n)}.$$

□

For $g = \left(\begin{smallmatrix} g_1 & \\ & g_2\end{smallmatrix}\right) \in M$, the action of g on $N \simeq \mathrm{M}_{k\times n-k}(F)$ is given by $g\cdot X = g_1 X g_2^{-1}$. The dual action of M on $\widehat{N}$ is given by:

$$g\cdot\psi_A(X) = \psi_A(g^{-1}\cdot X) = \psi_0(\mathrm{Tr}(g_2 A^\top g_1^{-1}X)) = \psi_{g_1^{-\top}Ag_2^\top}(X),$$

i.e., $g\cdot\psi_A = \psi_{g_1^{-\top}Ag_2^\top}$. Clearly, the M-orbits on $\widehat{N}$ are represented by $\psi_r = \psi_{A_r}$ where

$$A_r = \begin{bmatrix} I_r & 0\\ 0 & 0\end{bmatrix} \in \mathrm{M}_{k\times n-k}(F), \quad 0 \le r \le \min\{k, n-k\}.$$

For ψ_r, the subgroup $M_r = M_{\psi_r}$ of M consists of those $\left(\begin{smallmatrix} g_1 & \\ & g_2\end{smallmatrix}\right) \in M$ such that $g_1^{-\top}A_r g_2^\top = A_r$. Writing $g_1^{-\top} = \left(\begin{smallmatrix} g_{11} & g_{12}\\ g_{13} & g_{14}\end{smallmatrix}\right)$ and $g_2^{-\top} = \left(\begin{smallmatrix} g_{21} & g_{22}\\ g_{23} & g_{24}\end{smallmatrix}\right)$ as block matrices corresponding to the partition $(r, k-r)$ of k, and $(r, n-k-r)$ of $n-k$, we need

$$\left(\begin{smallmatrix} g_{11} & 0\\ g_{13} & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix} g_{11} & g_{12}\\ g_{13} & g_{14}\end{smallmatrix}\right)\left(\begin{smallmatrix} I_r & 0\\ 0 & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix} I_r & 0\\ 0 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} g_{21} & g_{22}\\ g_{23} & g_{24}\end{smallmatrix}\right) = \left(\begin{smallmatrix} g_{21} & g_{22}\\ 0 & 0\end{smallmatrix}\right).$$

This is equivalent to $g_{13} = 0, g_{22} = 0$, and $g_{11} = g_{21}$. Therefore,

$$M_r = \left\{ \left(\begin{smallmatrix} h_1 & 0 & 0 & 0\\ h_5 & h_4 & 0 & 0\\ 0 & 0 & h_1 & h_3\\ 0 & 0 & 0 & h_2\end{smallmatrix}\right) : h_1 \in G_r, h_4 \in G_{k-r}, h_2 \in G_{n-k-r},\right.$$
$$\left. h_5 \in \mathrm{M}_{k-r\times r}(F),\ h_3 \in \mathrm{M}_{r\times n-k-r}(F) \right\}. \quad (2.2)$$

2.3. **The twisted Jacquet module π_{N,ψ_k} in the case $n = 2k$.**
We first record some facts that we need about the characters of the Steinberg representation of G_n ([Car85]) and the representation $\mathrm{Ind}_{F_n^\times}^{G_n}\theta$ where θ is a character of $F_n^\times$. The following result is from O. Gorodetsky and Z. Hazan [GH19].

Lemma 2.4. [GH19, eq 6.2] *Let St_{G_n} denote the Steinberg representation of G_n. We use the same symbol St_{G_n} for the character of St_{G_n}. If g is non-semisimple, then $\mathrm{St}_{G_n}(g) = 0$. If g is semisimple and comes from $F_n^\times$ with minimal polynomial $p(X) \in F[X]$ of degree $m|n$, we have*

$$\mathrm{St}_{G_n}(g) = (-1)^{n-n'}q'^{\binom{n'}{2}}, \quad \text{where } n' = n/m,\ q' = q^m.$$

The following result is from [Pra00]:

Lemma 2.5. [Pra00, Lemma 1] *For a character θ of $F_k^\times$, the character $\chi_{\mathrm{Ind}_{F_k^\times}^{G_k}\theta}$ of the induced representation $\mathrm{Ind}_{F_k^\times}^{G_k}\theta$ equals 0 unless g is semisimple and comes from $F_k^\times$. In case g comes from $F_k^\times$ and $F[\lambda]=F_m$ for an eigenvalue λ of g, we have*

$$\chi_{\mathrm{Ind}_{F_k^\times}^{G_k}\theta}(g)=q'^{\binom{k'}{2}}\left(\sum_{i=0}^{m-1}\theta(\phi^i(\lambda))\right)\prod_{j=1}^{k'-1}(q'^j-1)\qquad \text{where } k'=k/m,\ q'=q^m.$$

Lemma 2.6. *Let θ be a character of $F_n^\times$, and let $\chi_{\mathrm{Ind}_{F_n^\times}^{G_n}\theta}$ denote the character of the representation $\mathrm{Ind}_{F_n^\times}^{G_n}\theta$. Let St_{G_n} be the Steinberg representation of G_n. Let Θ_θ be the class function as in Definition 2.2.*

(1) $\chi_{\mathrm{Ind}_{F_n^\times}^{G_n}\theta}(g)=\mathrm{St}_{G_n}(g)\cdot\Theta_\theta(g)$.

(2) *If θ is a regular character of $F_n^\times$ and π_θ, the associated cuspidal representation of G_n, then*

$$\mathrm{Ind}_{F_n^\times}^{G_n}\theta\simeq\mathrm{St}_{G_n}\otimes\pi_\theta.$$

Proof. Part (2) clearly follows from part (1). It suffices to prove $\chi_{\mathrm{Ind}_{F_n^\times}^{G_n}\theta}(g)=\mathrm{St}_{G_n}(g)\cdot\Theta_\theta(g)$. We can assume g is semisimple and comes from $F_n^\times$, for otherwise both $\Theta_\theta(g)\,\mathrm{St}_{G_n}(g)$ and $\chi_{\mathrm{Ind}_{F_n^\times}^{G_n}}(g)$ are zero. In the case when g comes from $F_n^\times$, it follows from Lemma 2.4 and Lemma 2.1 that $\Theta_\theta(g)\,\mathrm{St}_{G_n}(g)$ equals the expression for $\chi_{\mathrm{Ind}_{F_n^\times}^{G_n}}(g)$ given in Lemma 2.5. ☐

In the case when $n=2k$ and $P=MN$ is the maximal parabolic subgroup of G_n corresponding to the partition (k,k) of $n=2k$, the group $M\simeq G_k\times G_k$ and N is isomorphic to the additive group $\mathrm{M}_{k\times k}(F)$. In the case when the character $\psi_k=\psi_A$ of N is non-degenerate (that is $\mathrm{Rank}(A)=k$), it follows from (2.2) that the group

$$M_{\psi_k}=\left\{\left(\begin{smallmatrix}h_1&\\&h_1\end{smallmatrix}\right):h_1\in G_k\right\}\simeq G_k.$$

The twisted Jacquet module π_{N,ψ_k} was determined by D. Prasad:

Theorem 2.7. [Pra00] *For a cuspidal representation $\pi=\pi_\theta$ of G_{2k} associated to a regular character θ of $F_{2k}^\times$, we have*

$$\pi_{N,\psi_k}\simeq\mathrm{Ind}_{F_k^\times}^{G_k}(\theta|_{F_k^\times})$$

as $M_{\psi_k}\simeq G_k$-modules.

In the work [GH19], O. Gorodetsky and Z. Hazan studied the following generalization of Theorem 2.7 to G_{kn}. Let $P=MN$ be the Levi decomposition of the standard parabolic subgroup P of G corresponding to the partition of kn into k parts each of size n. The Levi subgroup M is isomorphic to $G_n\times\cdots\times G_n$, and the unipotent radical N consists of block upper triangular matrices where each block has size $n\times n$ and the diagonal blocks are all I_n. The non-degenerate characters of N form a single orbit under the action of M on the characters of N. For a cuspidal representation π_θ of G_{kn} associated to a regular character θ of $F_{kn}^\times$, and a non-degenerate character ψ of N, the structure of the module $\pi_{N,\psi}$ is:

Theorem 2.8. [GH19, Theorem 1] *For $k \geq 2$, let π_θ be a cuspidal representation of G_{kn} associated to a regular character θ of $F_{kn}^\times$ and a non-degenerate character ψ of the unipotent radical N of P as above. In this case $M_\psi \simeq G_n$ and*

$$\pi_{N,\psi} \simeq \mathrm{St}_{G_n}^{\otimes(k-2)} \otimes \mathrm{Ind}_{F_n^\times}^{G_n}(\theta|_{F_n^\times})$$

as an M_ψ-module.

The version of Theorem 1 as it appears in [GH19] is that

$$\pi_{N,\psi} \simeq \mathrm{St}_{G_n}^{\otimes(k-1)} \otimes \pi_{\theta|_{F_n^\times}},$$

which assumes the condition that the restriction $\theta|_{F_n^\times}$ is regular, so that the cuspidal representation $\pi_{\theta|_{F_n^\times}}$ of G_n is defined. The version we stated above is implicit in the results of [GH19] (Theorem 3, Equation 6.2, and Lemma 2.3), and has been confirmed to be correct in private communication with O. Gorodetsky and Z. Hazan. We note that in the special case when $\theta|_{F_n^\times}$ is regular, both the versions agree, as can be seen from Lemma 2.6.

3. Some combinatorial results

We first record some results that we will need from [BKK25]. Let

$$\mathrm{M}_{m\times n}(r, F) = \{X \in \mathrm{M}_{m\times n}(F)\colon \mathrm{Rank}(X) = r\},$$

and let $a(m \times n, r, q)$ denote the cardinality of $\mathrm{M}_{m\times n}(r, F)$. It is not hard to see that

$$a(m \times n, r, q) = {\textstyle\begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q} |G_r|,$$

where ${\textstyle\begin{bmatrix} m \\ r \end{bmatrix}_q}$ is the Gaussian binomial coefficient (see (3.3)). For $\alpha \in F$, let

$$f^\alpha_{m\times n,r,k}(q) = \#\{X \in \mathrm{M}_{m\times n}(r, F)\colon X_{11} + \cdots + X_{kk} = \alpha\}. \tag{3.1}$$

Theorem 3.1. [BKK25, Theorem 4.2] *Let ψ_0 be a non-trivial additive character of F, and let $A \in \mathrm{M}_{m\times n}(k, F)$. The sum*

$$\sum_{X\in \mathrm{M}_{m\times n}(r,F)} \psi_0(\mathrm{Tr}(A^\top X)),$$

as a function of A, only depends on $k = \mathrm{Rank}(A)$ and equals the difference

$$g_{m\times n,r,k}(q) := f^0_{m\times n,r,k}(q) - f^1_{m\times n,r,k}(q).$$

We have:

$$g_{m\times n,r,k}(q) = \sum_{i=0}^{r} (-1)^i {\textstyle\begin{bmatrix} k \\ i \end{bmatrix}_q} q^{\binom{i}{2}+k(r-i)}\, a((m-k)\times(n-k), r-i, q). \tag{3.2}$$

We recall the definitions of the q-binomial coefficient, the q-Pochhammer symbol, and the q-binomial theorem.

Definition 3.2. Let n be a non-negative integer. Let $(T; q)_n$ be the polynomial in $\mathbb{Z}[T]$ of degree n defined by

$$(T; q)_n = \prod_{i=0}^{n-1}(1 - Tq^i), \quad n > 0.$$

We take $(T;q)_0 = 1$. The Gaussian or q-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}_q$ counts the number of r dimensional vector subspaces of F^n and is given by the formula

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{(q^{n-r+1};q)_r}{(q;q)_r}. \tag{3.3}$$

The following polynomial identity in $\mathbb{Z}[T]$ is known as the q-binomial theorem:

$$(T;q)_n = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q (-1)^r q^{\binom{r}{2}} T^r. \tag{3.4}$$

Consider the $\mathbb{Q}$-vector space consisting of polynomials in $\mathbb{Q}[T]$ of degree at most n. The polynomials $(1, T, T^2, \ldots, T^n)$ and $((T;q)_0, \ldots, (T;q)_n)$ are two bases for this vector space. The expansion of the second basis vectors in terms of the first basis is given by the q-binomial theorem (3.4). The next result gives the expansion of the first basis vectors in terms of the second basis:

Lemma 3.3.

$$q^{\binom{n}{2}} T^n = \sum_{i=0}^{n} (-1)^i q^{\binom{n-i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q (T;q)_i \tag{3.5}$$

Proof. The RHS of (3.5) can be written using the q-binomial theorem mentioned above as

$$\sum_{i=0}^{n} (-1)^i q^{\binom{n-i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q \Big(\sum_{k=0}^{i} q^{\binom{k}{2}} \begin{bmatrix} i \\ k \end{bmatrix}_q (-T)^k\Big)$$

Interchanging the order of summation, and using $\begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} i \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ n-i \end{bmatrix}_q$, we get

$$(-1)^n \sum_{k=0}^{n} q^{\binom{k}{2}} (-T)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \Big(\sum_{i \geq k} (-1)^{n-i} q^{\binom{n-i}{2}} \begin{bmatrix} n-k \\ n-i \end{bmatrix}_q\Big).$$

The expression $\displaystyle\sum_{i \geq k} (-1)^{n-i} q^{\binom{n-i}{2}} \begin{bmatrix} n-k \\ n-i \end{bmatrix}_q$ equals $(1;q)_{n-k} = \delta_{k,n}$. Thus, the above expression is just $q^{\binom{n}{2}} T^n$. □

Lemma 3.4. *The following polynomial identity holds in the polynomial ring* $\mathbb{Q}[S]$*:*

$$\sum_r a(m \times n, r, q)(S;q)_{n-r} = q^{mn}(Sq^{-m};q)_n.$$

Proof. For $\ell \geq n$, the size of the set $\mathrm{M}_{n\times\ell}(n, F)$ of all full rank $n \times \ell$ matrices is

$$a(n \times \ell, n, q) = \prod_{i=0}^{n-1} (q^\ell - q^i).$$

For such a matrix $X \in \mathrm{M}_{n\times\ell}(n, F)$, and for $m \leq \ell$, let $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ where X_1 has size $n \times m$ and X_2 has size $n \times (\ell - m)$. For X_1 of rank r, let U denote the column span of X_1. The number of matrices X_2 such that $\begin{bmatrix} X_1 & X_2 \end{bmatrix}$ has rank n, is same as the number of maps $F^{\ell-m} \to F^n$ such that the associated map $F^{\ell-m} \to F^n/U$ has rank $(n - r)$. In other words, the number of choices for X_2 is

$$q^{(\ell-m)r} \cdot a(n-r \times \ell-m, n-r, q) = q^{(\ell-m)r} \cdot \prod_{i=0}^{n-r-1} (q^{\ell-m} - q^i).$$

Therefore, we get the identity

$$\prod_{i=0}^{n-1}(q^{\ell}-q^{i})=\sum_{r}a(m\times n,r,q)q^{(\ell-m)r}\prod_{i=0}^{n-r-1}(q^{\ell-m}-q^{i}).$$

Multiplying by q^{mn} we can rewrite this as

$$q^{mn}\prod_{i=0}^{n-1}(q^{\ell}-q^{i})=\sum_{r}a(m\times n,r,q)q^{\ell r}\prod_{i=0}^{n-r-1}(q^{\ell}-q^{i+m}).$$

This implies the polynomial identity in $\mathbb{Q}[T]$:

$$q^{mn}\prod_{i=0}^{n-1}(T-q^{i})=\sum_{r}a(m\times n,r,q)T^{r}\prod_{i=0}^{n-r-1}(T-q^{i+m}),$$

because the difference between the RHS and LHS is a polynomial of degree at most n in $\mathbb{Q}[T]$ with infinitely many roots $T=q^{\ell}$ for $\ell\geq n$, and hence it is the zero polynomial. Dividing by T^{n}, we get the identity in $\mathbb{Q}[T^{-1}]$:

$$q^{mn}\prod_{i=0}^{n-1}(1-q^{i}T^{-1})=\sum_{r}a(m\times n,r,q)\prod_{i=0}^{n-r-1}(1-q^{i+m}T^{-1}).$$

In terms of $S=q^{m}T^{-1}$ we can rewrite this as the identity in $\mathbb{Q}[S]$ given by:

$$q^{mn}(Sq^{-m};q)_{n}=\sum_{r}a(m\times n,r,q)\,(S;q)_{n-r}.$$

□

4. A FUNDAMENTAL REPRESENTATION OF $P_{k,n}$.

For a representation τ of $P=P_{k,n}$, we recall from §2.2, that the twisted Jacquet module τ_r of τ is a representation of the group $M_r\subset P^{*}_{r,k}\times P_{r,n-k}$ given by

$$M_r=\left\{\left(\begin{smallmatrix}g_1&\\&g_2\end{smallmatrix}\right):g_1=\left(\begin{smallmatrix}h_1&0\\h_5&h_4\end{smallmatrix}\right)\in P^{*}_{r,k},\ g_2=\left(\begin{smallmatrix}h_1&h_3\\0&h_2\end{smallmatrix}\right)\in P_{r,n-k}\right\},$$

(we note that the same matrix $h_1\in G_r$ occurs in both blocks g_1 and g_2). The group M_r is trivial if $r>\mu:=\min\{k,n-k\}$.

4.1. Equivalence between the Categories $\mathrm{Rep}(P)$ and $\oplus_{r=0}^{\mu}\mathrm{Rep}(M_r)$.

Let $P=P_{k,n}$ and let $M_0,\dots,M_\mu$ be the groups mentioned above. Let $\oplus_{r=0}^{\mu}\mathrm{Rep}(M_r)$ be the direct sum of the (Abelian)-categories $\mathrm{Rep}(M_0),\dots,\mathrm{Rep}(M_\mu)$. Consider the twisted Jacquet functor

$$\mathfrak{F}:\mathrm{Rep}(P)\to\bigoplus_{r=0}^{\mu}\mathrm{Rep}(M_r),\quad \mathfrak{F}(\tau)=(\tau_0,\dots,\tau_\mu).$$

Let $P_r=N\rtimes M_r$. The character ψ_r of N extends to a well-defined character of P_r by $\psi_r(mn)=\psi_r(n)$ (which we denote by the same symbol ψ_r). Any representation ρ_r of M_r inflates to a representation of $P_r=N\rtimes M_r$, which we continue to denote by ρ_r. We also consider the functor

$$\mathfrak{G}:\bigoplus_{r=0}^{\mu}\mathrm{Rep}(M_r)\to\mathrm{Rep}(P),\quad \mathfrak{G}(\rho_0,\dots,\rho_\mu)=\oplus_{r=0}^{\mu}\,\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\rho_r),$$

defined in terms of the induction functors $\mathrm{Ind}_{P_r}^{P}:\mathrm{Rep}(P_r)\to\mathrm{Rep}(P)$ and the inflation functors $\mathrm{Rep}(M_r)\to\mathrm{Rep}(P_r)$. We claim that the functors $\mathfrak{F}$ and $\mathfrak{G}$ yield an equivalence between the categories $\mathrm{Rep}(P)$ and $\oplus_{r=0}^{\mu}\mathrm{Rep}(M_r)$. In other words, we must show that the compositions $\mathfrak{G}\circ\mathfrak{F}$ and $\mathfrak{F}\circ\mathfrak{G}$ are naturally isomorphic to the identity functors of $\mathrm{Rep}(P)$ and $\oplus_{r=0}^{\mu}\mathrm{Rep}(M_r)$ respectively.

The assertion that $\mathfrak{G}\circ\mathfrak{F}$ is naturally isomorphic to the identity functor of $\mathrm{Rep}(P)$, is to say that τ is isomorphic to the representation $\oplus_{r=0}^{\mu}\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\tau_r)$, which is proved in the next theorem.

Theorem 4.1. *Let P be a maximal parabolic subgroup of G_n associated to the partition $(k, n-k)$ of n. Let $\mu = \min\{k, n-k\}$. For a representation τ of P as above, we have*

$$\tau \simeq \bigoplus_{r=0}^{\mu}\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\tau_r).$$

Proof. We recall that by the *little group method* for representations of a group with an abelian normal subgroup (for example [CSST09, Theorem 5.2]), any irreducible representation of P is of the form $\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\eta)$ for some $0\le r\le\mu$ and some irreducible representation η of M_r. Therefore, we may write

$$\tau \simeq \bigoplus_{r=0}^{\mu}\bigoplus_{\eta\in\widehat{M_r}} a_{r,\eta}\,\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\eta), \tag{4.1}$$

for some non-negative integers $a_{r,\eta}$. In order to show that $\tau\simeq\oplus_{r=0}^{\mu}\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\tau_r)$, it is enough to show that $\tau_r\simeq\oplus_{\eta\in\widehat{M_r}}a_{r,\eta}\,\eta$, or equivalently $\langle\chi_{\tau_r},\chi_\eta\rangle_{M_r}=a_{r,\eta}$. For any $\eta\in\widehat{M_r}$, it follows from Proposition 2.3 that

$$\langle\chi_{\tau_r},\chi_\eta\rangle_{M_r}=\langle\chi_{\tau|_{P_r}},\psi_r\chi_\eta\rangle_{P_r}=\langle\chi_\tau,\chi_{\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\eta)}\rangle_P.$$

Since $\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\eta)$ is irreducible for each irreducible representation η of M_r, it follows that

$$\langle\chi_{\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\eta)},\chi_{\mathrm{Ind}_{P_{r'}}^{P}(\psi_{r'}\otimes\eta')}\rangle_P=\begin{cases}1 & \text{if } r=r' \text{ and } \eta=\eta'\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, substituting (4.1) for τ in $\langle\chi_\tau,\chi_{\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\eta)}\rangle_P$, we just get $a_{r,\eta}$. This establishes that $\langle\chi_{\tau_r},\chi_\eta\rangle_{M_r}=a_{r,\eta}$. □

The assertion that $\mathfrak{F}\circ\mathfrak{G}$ is naturally isomorphic to the identity functor of $\oplus_{r=0}^{\mu}\mathrm{Rep}(M_r)$, is to say that if we start with $(\rho_0,\ldots,\rho_\mu)\in\oplus_{r=0}^{\mu}\mathrm{Rep}(M_r)$, then the r-th twisted Jacquet module τ_r of $\tau:=\oplus_{r=0}^{\mu}\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\rho_r)$ is isomorphic to ρ_r. The proof of this assertion is similar: expanding $\rho_r\simeq\oplus_{\eta\in\widehat{M_r}}b_{r,\eta}\,\eta$ and $\tau_r=\sum_{\eta\in\widehat{M_r}}a_{r,\eta}\,\eta$, we must show that $a_{r,\eta}=b_{r,\eta}$. By the above theorem, we know $\tau\simeq\oplus_{r=0}^{\mu}\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\tau_r)$ and hence

$$\bigoplus_{r=0}^{\mu}\bigoplus_{\eta\in\widehat{M_r}}a_{r,\eta}\,\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\eta)=\tau=\bigoplus_{r=0}^{\mu}\bigoplus_{\eta\in\widehat{M_r}}b_{r,\eta}\,\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\eta).$$

Since $\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\eta)$ is irreducible, $\mathrm{Ind}_{P_r}^{P}(\psi_r\otimes\eta)\simeq\mathrm{Ind}_{P_{r'}}^{P}(\psi_{r'}\otimes\eta')$ if and only if $r=r'$ and $\eta=\eta'$. It follows that $a_{r,\eta}=b_{r,\eta}$ for all r and η.

Corollary 4.2. *A representation τ of the maximal parabolic subgroup $P = P_{k,n}$ of G_n is irreducible if and only if all but one of the twisted Jacquet modules $\tau_0, \dots, \tau_\mu$ are zero, and the non-zero one is irreducible.*

Proof. This is clear from the equivalence between the categories $\oplus_{r=0}^{\mu} \mathrm{Rep}(M_r)$ and $\mathrm{Rep}(P)$. $\square$

We illustrate the theorem with the case $k = n-1$. Here the parabolic subgroup $P_{n-1,n} = F^\times \times \mathrm{Mir}_n$, where $F^\times$ is the subgroup of scalar matrices, and

$$\mathrm{Mir}_n = \{ \left(\begin{smallmatrix} h & v \\ & 1 \end{smallmatrix}\right) : h \in G_{n-1}, v \in \mathrm{M}_{n-1 \times 1}(F) \}, \tag{4.2}$$

is the Mirabolic subgroup of G_n. Any representation of $P_{n-1,n}$ is of the form $\theta \boxtimes \pi$ where θ is a character of $F^\times$, and π is a representation of Mir_n, and hence it suffices to study representations $\tau = 1 \boxtimes \pi$. The Jacquet modules $\tau_0 = 1 \boxtimes \pi_0$ and $\tau_1 = 1 \boxtimes \pi_1$ are representations of $M_0 = F^\times \times G_{n-1}$ and $M_1 = F^\times \times \mathrm{Mir}_{n-1}$ respectively, where G_{n-1} and Mir_{n-1} are subgroups of Mir_n given by

$$G_{n-1} = \{ \left(\begin{smallmatrix} h & \\ & 1 \end{smallmatrix}\right) \in \mathrm{Mir}_n \colon h \in G_{n-1} \}, \quad \mathrm{Mir}_{n-1} = \{ \left(\begin{smallmatrix} h & \\ & 1 \end{smallmatrix}\right) \in \mathrm{Mir}_n \colon h \in \mathrm{Mir}_{n-1} \}. \tag{4.3}$$

We recall the Bernstein-Zelevinsky functors $\Psi^-, \Psi^+, \Phi^-, \Phi^+$ (see [BZ76, Section 5.11]). The assignment $\pi \mapsto \pi_0$ and $\pi \mapsto \pi_1$ are the functors Ψ^- and Φ^- respectively. The induction $\mathrm{Ind}_{P_0}^{P}(\tau_0)$ equals $1 \boxtimes \tilde{\pi_0}$ where $\tilde{\pi_0}$ is the inflation of π_0 from G_{n-1} to Mir_n, and the assignment $\pi_0 \mapsto \tilde{\pi_0}$ is the functor Ψ^+. The induction $\mathrm{Ind}_{P_1}^{P}(\psi_1 \otimes \tau_1)$ equals $1 \boxtimes \mathrm{Ind}_{\mathrm{Mir}_{n-1} \ltimes F^{n-1}}^{\mathrm{Mir}_n}(\psi_1 \otimes \pi_1)$, and the assignment $\pi_1 \mapsto \mathrm{Ind}_{\mathrm{Mir}_{n-1} \ltimes F^{n-1}}^{\mathrm{Mir}_n}(\psi_1 \otimes \pi_1)$ is the Φ^+ functor. Theorem 4.1, asserts that

$$\pi \simeq \Psi^+ \Psi^-(\pi) \oplus \Phi^+ \Phi^-(\pi),$$

which corresponds to the exact sequence in part d) of [BZ76, Proposition 5.12]. We also illustrate Corollary 4.2 for $P_{n-1,n}$. The corollary asserts that any irreducible representation of $P_{n-1,n}$ is of the form $\theta \boxtimes \pi$, where θ is a character of $F^\times$, and the representation π of Mir_n is either of the form $\Psi^+(\pi_0)$ for an irreducible representation π_0 of G_{n-1}, or it is $\Phi^+(\pi_1)$ for an irreducible representation π_1 of Mir_{n-1}. In the latter case, again applying the corollary to the representation $1 \boxtimes \pi_1$ of $P_{n-2,n-1}$, and repeating if necessary, we conclude that π must be of the form $(\Phi^+)^{n-r-1} \Psi^+(\pi_r)$ where π_r is an irreducible representation of G_r for some $0 \le r \le n-1$ ([BZ76, Corollary 5.13]).

4.2. **Construction of $\Pi_{k,n}$.** In this section, we construct a representation $\Pi_{k,n}$ of the parabolic group $P_{k,n}$ for $1 \le k \le n$, which will satisfy a universal property with respect to restrictions to $P_{k,n}$ of cuspidal representations of G_n (see Theorem 4.13). We take the representation $\Pi_{n,n}$ of $P_{n,n} = G_n$ to be the trivial representation. By Theorem 4.1, a representation τ of $P_{k,n}$ is uniquely determined by its twisted Jacquet modules τ_r for $0 \le r \le \min\{k, n-k\}$. For $g = \left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right) \in M_r$ with $g_1 = \left(\begin{smallmatrix} h_1 & 0 \\ h_5 & h_4 \end{smallmatrix}\right)$ and $g_2 = \left(\begin{smallmatrix} h_1 & h_3 \\ 0 & h_2 \end{smallmatrix}\right)$, let

$$M_r \xrightarrow{p_2} P_{r,n-k}, \quad M_r \xrightarrow{p_3} G_r,$$

be the epimorphisms defined by

$$p_2(g) = g_2, \quad p_3(g) = h_1.$$

Any representation of $P_{r,n-k}$ can be inflated to a representation of M_r via p_2. Let St_{G_r} be the representation of $P_{r,n-k}$ obtained by inflating the Steinberg representation of G_r via p_3. We define $\Pi_{k,n}$ recursively for $1 \leq k \leq n-1$ by requiring that its twisted Jacquet module $(\Pi_{k,n})_r$ is zero for $r = 0$, and it equals $\mathrm{St}_{G_r} \otimes \Pi_{r,n-k}$ (inflated to M_r as above) for $1 \leq r \leq \min\{k, n-k\}$:

Definition 4.3. The representation $\Pi_{k,n}$ of $P_{k,n}$ for $1 \leq k \leq n$ is the trivial representation of $P_{n,n}$ in case $k = n$, and in case $1 \leq k \leq n-1$, it is defined by

$$\Pi_{k,n} = \sum_{r=1}^{\min\{k,n-k\}} \mathrm{Ind}_{P_{k,n}(r)}^{P_{k,n}} (\psi_r \otimes [\mathrm{St}_{G_r} \otimes \Pi_{r,n-k}])$$

where $P_{k,n}(r) = M_r \ltimes N_{k,n}, N_{k,n} \simeq \mathrm{M}_{k \times n-k}(F)$.

The recursion terminates in exactly $(n-k)$ steps. To see this, we induct on the 'length' $\mathrm{len}(\Pi_{k,n}) := n-k$. If $\mathrm{len}(\Pi_{k,n}) = 0$ i.e. $n = k$, we have $\Pi_{n,n} = 1_{P_{n,n}}$ and we do not use the recursion. For $\Pi_{k,n}$ with $(n-k) = \ell$, we assume inductively that that the recursion for $\Pi_{i,j}$ terminates in $(j-i)$ steps if $(j-i) < \ell$. It follows that the representations $\Pi_{r,n-k}$ appearing in the first recursion for $\Pi_{k,n}$, in turn require $(n-k-r)$ steps, because $r \geq 1$ and hence $n-k-r = \ell - r < \ell$. In particular, the recursion for $\Pi_{1,n-k}$ requires $\ell - 1 = (n-k-1)$ steps. Therefore, we conclude that the recursion for $\Pi_{k,n}$ terminates in $(n-k)$ steps.

The simplest example of $\Pi_{k,n}$ after $\Pi_{n,n}$, is $\Pi_{n-1,n}$. We use the notation from the discussion after Theorem 4.1.

Lemma 4.4. *The representation* $\Pi_{n-1,n}$ *of* $P_{n-1,n} = F^\times \times \mathrm{Mir}_n$ *has dimension* $(q^{n-1}-1)$ *and equals* $1_{F^\times} \boxtimes \Phi^+(1_{\mathrm{Mir}_{n-1}})$.

Proof. By definition 4.3, we have $\Pi_{n-1,n} = \mathrm{Ind}_{P_{n-1,n}(1)}^{P_{n-1,n}}(\psi_1 \otimes [\mathrm{St}_{G_1} \otimes \Pi_{1,1}])$. Since $\mathrm{St}_{G_1} = 1_{G_1}$ and $\Pi_{1,1} = 1_{P_{1,1}}$ are trivial representations, it follows that $[\mathrm{St}_{G_1} \otimes \Pi_{1,1}]$, viewed as a representation of $P_{n-1,n}(1)$, is the trivial representation. We recall that $P_{n-1,n} = F^\times \times \mathrm{Mir}_n$ and $P_{n-1,n}(1) = F^\times \times (\mathrm{Mir}_{n-1} \ltimes F^{n-1})$, where $F^\times$ is the group of scalar matrices in G_n, and

$$\begin{aligned} \mathrm{Mir}_n &= \{\left(\begin{smallmatrix} h & v \\ & 1 \end{smallmatrix}\right) \in P_{n-1,n} \colon h \in G_{n-1}, v \in F^{n-1}\}, \\ \mathrm{Mir}_{n-1} \ltimes F^{n-1} &= \{\left(\begin{smallmatrix} h & v \\ & 1 \end{smallmatrix}\right) \in \mathrm{Mir}_n \colon h \in \mathrm{Mir}_{n-1}\}. \end{aligned}$$

Therefore,

$$\Pi_{n-1,n} = 1_{F^\times} \boxtimes \mathrm{Ind}_{\mathrm{Mir}_{n-1} \ltimes F^{n-1}}^{\mathrm{Mir}_n}(\psi_1 \otimes 1_{\mathrm{Mir}_{n-1}}).$$

In particular,

$$\dim(\Pi_{n-1,n}) = |\,\mathrm{Mir}_n\,| / |\,\mathrm{Mir}_{n-1} \ltimes F^{n-1}| = (q^{n-1}-1).$$

For a representation τ of Mir_{n-1}, we recall that $\Phi^+(\tau)$ is the representation of Mir_n given by $\mathrm{Ind}_{\mathrm{Mir}_{n-1} \ltimes F^{n-1}}^{\mathrm{Mir}_n}(\psi_1 \otimes \tau)$. Thus, we conclude that $\Pi_{n-1,n} = 1_{F^\times} \boxtimes \Phi^+(1_{\mathrm{Mir}_{n-1}})$. $\square$

The representation $\Pi_{1,n}$ of $P_{1,n}$ can also be described in terms of the Φ^+ functor. We start with a definition:

Definition 4.5. Let $\dagger : G_n \to G_n$ be the involutive automorphism given by

$$g^\dagger = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix} g^{-\top} \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}.$$

For a subgroup H of G_n, we denote $\dagger(H)$ by $H^\dagger$, and for a representation π of H, the representation $\pi \circ \dagger$ of $H^\dagger$ will be denoted $\pi^\dagger$. We note that $P_{k,n}^\dagger = P_{n-k,n}$ for $1 \leq k \leq n-1$. For a cuspidal representation π_θ of G_n, we claim that $\pi_\theta^\dagger$ is isomorphic to $\pi_{\bar{\theta}}$ where $\bar{\theta}$ is the complex conjugate of the regular character θ of $F_n^\times$, i.e.,

$$\pi_\theta^\dagger \simeq \pi_{\bar{\theta}}. \tag{4.4}$$

To see this, we note that the character $\chi_{\pi_\theta^\dagger}(g) = \chi_{\pi_\theta}(g^{-\top}) = \Theta_\theta(g^{-\top})$ because $g^\dagger$ is conjugate to $g^{-\top}$ by definition. Since $g^\top$ is conjugate to g for all $g \in \mathrm{GL}(n, K)$ (for an arbitrary field K), it follows that $\chi_{\pi_\theta^\dagger}(g) = \Theta_\theta(g^{-1})$. It follows from Theorem 2.1, that $\Theta_\theta(g^{-1}) = \Theta_{\bar{\theta}}(g)$, because the eigenvalues of g^{-1} are the reciprocals of the eigenvalues of g, and $\theta(\phi^i(\lambda^{-1})) = \bar{\theta}(\phi^i(\lambda))$. Therefore, $\pi_\theta^\dagger$ is isomorphic to $\pi_{\bar{\theta}}$.

Since $P_{n-1,n} = F^\times \times \mathrm{Mir}_n$, we get $P_{1,n} = P_{n-1,n}^\dagger = F^\times \times \mathrm{Mir}_n^\dagger$, where $F^\times$ is the group of scalar matrices in G_n, and

$$\mathrm{Mir}_n^\dagger = \{\left(\begin{smallmatrix} 1 & v \\ & h \end{smallmatrix}\right) : v \in M_{1\times n-1}(F), h \in G_{n-1}\}.$$

Similarly,

$$P_{1,n}(1) = F^\times \times (\mathrm{Mir}_{n-1} \ltimes F^{n-1})^\dagger.$$

Lemma 4.6. *The representations* $\Pi_{1,n}$ *of* $P_{1,n} = F^\times \times \mathrm{Mir}_n^\dagger$, *and* $\Pi_{1,n}^\dagger$ *of* $P_{n-1,n} = F^\times \times \mathrm{Mir}_n$ *have dimension* $(q-1)(q^2-1)\cdots(q^{n-1}-1)$ *and are isomorphic to:*

1. $\Pi_{1,n}^\dagger = 1_{F^\times} \boxtimes (\Phi^+)^{n-1}(1_{\mathrm{Mir}_1})$.
2. $\Pi_{1,n} = 1_{F^\times} \boxtimes \mathrm{Ind}_{U_n}^{\mathrm{Mir}_n^\dagger}(\vartheta)$, *where* U_n *is the unipotent subgroup of* $\mathrm{Mir}_n^\dagger$ *consisting of upper triangular matrices with ones on the diagonal, and* $\vartheta(g) = \psi_0(g_{1,2} + \cdots + g_{n-1,n})$ *with* ψ_0 *a non-trivial additive character of* F.

Proof. (1) We prove this by induction on n starting with the base case $n = 2$. By Lemma 4.4 we have

$$\Pi_{1,2} = 1_{F^\times} \boxtimes \mathrm{Ind}_{\mathrm{Mir}_1 \ltimes F}^{\mathrm{Mir}_2}(\psi_1 \otimes 1_{\mathrm{Mir}_1}) = \mathrm{Ind}_{F^\times \times (\mathrm{Mir}_1 \ltimes F)}^{F^\times \times \mathrm{Mir}_2}(1_{F^\times} \boxtimes \psi_1).$$

We note that the subgroup of G_2 given by

$$F^\times \times (\mathrm{Mir}_1 \ltimes F) = \{\left(\begin{smallmatrix} a & b \\ & a \end{smallmatrix}\right) : a \in F^\times, b \in F\},$$

is preserved by the $\dagger$ involution. Also, $F^\times \times \mathrm{Mir}_2 = P_{1,2}$ and hence $(F^\times \times \mathrm{Mir}_2)^\dagger = P_{1,2}^\dagger = P_{1,2}$. Therefore,

$$\Pi_{1,2}^\dagger = \mathrm{Ind}_{F^\times \times (\mathrm{Mir}_1 \ltimes F)}^{P_{1,2}}(1_{F^\times} \boxtimes \bar{\psi_1}) = 1_{F^\times} \boxtimes \mathrm{Ind}_{\mathrm{Mir}_1 \ltimes F}^{\mathrm{Mir}_2}(\bar{\psi_1}) \simeq 1_{F^\times} \boxtimes \Phi^+(1_{\mathrm{Mir}_1}).$$

We assume inductively that $\Pi_{1,n-1}^\dagger = 1_{F^\times} \boxtimes (\Phi^+)^{n-2}(1_{\mathrm{Mir}_1})$. By definition, we have $\Pi_{1,n} = \mathrm{Ind}_{P_{1,n}(1)}^{P_{1,n}}(\psi_1 \otimes \Pi_{1,n-1})$. Using $P_{1,n} = F^\times \times \mathrm{Mir}_n^\dagger$ and $P_{1,n}(1)^\dagger =$

$F^\times \times (\mathrm{Mir}_{n-1} \ltimes F^{n-1})$, we get:

$$\begin{aligned}
\Pi^\dagger_{1,n} &= \mathrm{Ind}^{F^\times \times \mathrm{Mir}_n}_{F^\times \times (\mathrm{Mir}_{n-1} \ltimes F^{n-1})}(1_{F^\times} \boxtimes (\bar{\psi}_1 \otimes (\Phi^+)^{n-2}(1_{\mathrm{Mir}_1}))) \\
&= 1_{F^\times} \boxtimes \mathrm{Ind}^{\mathrm{Mir}_n}_{\mathrm{Mir}_{n-1} \ltimes F^{n-1}}(\bar{\psi}_1 \otimes (\Phi^+)^{n-2}(1_{\mathrm{Mir}_1})) \\
&\simeq 1_{F^\times} \boxtimes (\Phi^+)^{n-1}(1_{\mathrm{Mir}_1}).
\end{aligned}$$

(2) It is well known (for example [BZ76, Corollary 5.13a]) that the representation $(\Phi^+)^{n-1}(1_{\mathrm{Mir}_1})$ of Mir_n is isomorphic to $\mathrm{Ind}^{\mathrm{Mir}_n}_{U_n}(\vartheta)$ where $\vartheta(g) = \tilde{\psi}_0(g_{1,2} + \cdots + g_{n-1,n})$, and $\tilde{\psi}_0$ is any non-trivial additive character of F. We take $\tilde{\psi}_0 = \bar{\psi}_0$. Therefore, we have

$$\Pi^\dagger_{1,n} = 1_{F^\times} \boxtimes \mathrm{Ind}^{\mathrm{Mir}_n}_{U_n}(\vartheta).$$

Since $U_n^\dagger = U_n$, we get

$$\Pi_{1,n} = 1_{F^\times} \boxtimes \mathrm{Ind}^{\mathrm{Mir}^\dagger_n}_{U_n}(\bar{\vartheta}),$$

where $(\bar{\vartheta})(g) = \psi_0(g_{1,2} + \cdots + g_{n-1,n})$. □

The next result gives the dimension of $\Pi_{k,n}$:

Lemma 4.7. $\dim \Pi_{k,n} = (q^k - 1)(q^{k+1} - 1) \cdots (q^{n-1} - 1) = \prod_{i=k}^{n-1}(q^i - 1)$.

Proof. If $k = n$ we take the above product to be trivial, so that $\dim(\Pi_{k,n}) = 1$. We prove the formula by induction on n. For the base case $n = 1$, we must have $k = 1$ and hence $\dim(\Pi_{1,1}) = 1$. For $n > 1$, the dimension of the right-side of the recursive formula for $(\Pi_{k,n})$ is

$$\sum_r \tfrac{|P_{k,n}|}{|P_{k,n}(r)|} \dim(\mathrm{St}_{G_r}) \dim(\Pi_{r,n-k}).$$

It is easily seen that

$$\tfrac{|P_{k,n}|}{|P_{k,n}(r)|} = \tfrac{|G_k \times G_{n-k}|}{|M_r|} = \tfrac{|G_k \times G_{n-k}|}{|P^*_{r,k} \times P_{r,n-k}|} \tfrac{|P^*_{r,k} \times P_{r,n-k}|}{|M_r|} = \left[\begin{smallmatrix} k \\ r \end{smallmatrix}\right]_q \left[\begin{smallmatrix} n-k \\ r \end{smallmatrix}\right]_q |G_r| = a(k \times n-k, r, q)$$

where, in the first equality we have used the fact that $P_{k,n}(r) = M_r \ltimes N_{k,n}$ and $P_{k,n} = (G_k \times G_{n-k}) \ltimes N_{k,n}$, and in the third equality we have used the fact that $|G_k|/|P^*_{r,k}|$ equals the number $\left[\begin{smallmatrix} k \\ k-r \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} k \\ r \end{smallmatrix}\right]_q$ of $(k-r)$-dimensional subspaces of F^k, and $|G_{n-k}|/|P_{r,n-k}|$ equals the number $\left[\begin{smallmatrix} n-k \\ r \end{smallmatrix}\right]_q$ of r-dimensional subspaces of F^{n-k}. Together with the inductive hypothesis $\dim(\Pi_{r,n-k}) = \prod_{i=r}^{n-k-1}(q^i - 1)$, and the fact that $\dim(\mathrm{St}_{G_r}) = q^{\binom{r}{2}}$, the recursive formula for $\dim(\Pi_{k,n})$ simplifies to

$$\sum_r a(k \times n-k, r, q) q^{\binom{r}{2}} \prod_{i=r}^{n-k-1} (q^i - 1) = q^{\binom{n-k}{2}} \sum_r a(k \times n-k, r, q)(q^{-n+k+1}; q)_{n-k-r}.$$

By Lemma 3.4, this equals

$$q^{\binom{n-k}{2}} q^{k(n-k)} (q^{-n+1}; q)_{n-k} = (q^k - 1)(q^{k+1} - 1) \cdots (q^{n-1} - 1).$$

□

Corollary 4.8. *The representation $\Pi_{k,n}$ of $P_{k,n}$ is irreducible if and only if $k \in \{1, n-1, n\}$.*

Proof. If $k = n$, then $\Pi_{k,n}$ is irreducible as it is the trivial representation. We now assume $1 \leq k \leq n-1$. Let $\mu = \min\{k, n-k\}$. The twisted Jacquet modules $(\tau_1, \ldots, \tau_\mu)$ of $\tau = \Pi_{k,n}$ are given by $\tau_r \simeq \mathrm{St}_{G_r} \otimes \Pi_{r,n-k}$. By Lemma 4.7, these modules are non-zero as their dimension is positive. Therefore, by Corollary 4.2, $\Pi_{k,n}$ is irreducible if and only if $\mu = 1$, i.e., $k \in \{1, n-1\}$, and τ_1 is irreducible. If $k = n-1$, we have $\tau_1 \simeq \Pi_{1,1}$ which is trivial and hence irreducible. As for $k = 1$, we prove that $\Pi_{1,n}$ is irreducible by induction on n. If $n = 1$, then $\Pi_{1,1}$ is trivial and hence irreducible. We assume inductively that $\Pi_{1,n-1}$ is irreducible. The twisted Jacquet module τ_1 for $\tau = \Pi_{1,n}$ is isomorphic to $\Pi_{1,n-1}$, which is irreducible by the inductive hypothesis. This completes the proof that $\Pi_{1,n}$ is irreducible. $\square$

In the next result, we determine the character of $\Pi_{k,n}$ only for the special type of matrices $g = \left(\begin{smallmatrix} g_1 & g_1 g_3 \\ 0 & g_2 \end{smallmatrix}\right)$ for which the semisimple part of $g_1 \in G_k$ comes from $F_k^\times$. We denote the character of $\Pi_{k,n}$ as $\Theta_{k,n}$.

Theorem 4.9. *Let $g = \left(\begin{smallmatrix} g_1 & g_1 g_3 \\ 0 & g_2 \end{smallmatrix}\right) \in P_{k,n}$. If the semisimple part of $g_1 \in G_k$ comes from $F_k^\times$, then the character $\Theta_{k,n}$ of $\Pi_{k,n}$ at g is zero, unless the semisimple part of g comes from $F_d^\times$ where $d = gcd\{k, n\}$, in which case*

$$\Theta_{k,n}(g) = (-1)^{n-k} ({q'}^{t(g_1)}; q')_{t(g)-t(g_1)},$$

where $F[\lambda] = F_m \subset F_d$ for any eigenvalue λ of g, $q' = q^m$ and $t(g) = \dim_{F_n} \ker(g - \lambda I_n)$ and $t(g_1) = \dim_{F_k} \ker(g_1 - \lambda I_k)$.

The proof of this result requires several arguments, and the next subsection 4.3 is devoted to the proof.

4.3. **Proof of Theorem 4.9.**

Throughout this proof let s_1, s_2 and s denote the semisimple parts of g_1, g_2 and g respectively. Since s_1 comes from $F_k^\times$, its minimal polynomial which we denote $p(X)$ is irreducible in $F[X]$ of degree $m|k$. Let $\lambda \in F_m$ denote an eigenvalue of g_1, so that $p(X)$ factorizes as $p(X) = \prod_{i=0}^{m-1}(X - \phi^i(\lambda))$ in $F_m[X]$. The symbols r', k', n' and q' stand for $r/m, k/m, n/m$ and q^m respectively.

We prove the assertion by induction on the 'length' $(n-k)$. If $n - k = 0$, the semisimple part of $g = g_1$ indeed comes from $F_d^\times$, and also $t(g) - t(g_1) = 0$. Therefore $(-1)^{n-k}({q'}^{k'}; q')_{t(g)-t(g_1)} = (-1)^0 (q^m; q^m)_0 = 1$, which agrees with the fact that $\Pi_{k,k}$ is the trivial representation of $P_{k,k} = G_k$. For $n - k > 0$, we assume inductively that the assertion is true for $\Pi_{r,n-k}$ which has 'length' $(n-k-r) < n-k$. First we prove the theorem for the case when s does not come from $F_n^\times$.

Step 1: Proof of Theorem 4.9 in case s does not come from $F_n^\times$

We start with the following elementary lemma:

Lemma 4.10. *Let $g = \left(\begin{smallmatrix} g_1 & g_1 g_3 \\ 0 & g_2 \end{smallmatrix}\right) \in P_{k,n}$ and let $g = s \cdot u$ be the Jordan decomposition of g into its semisimple and unipotent parts. If $s = \left(\begin{smallmatrix} s_1 & s_3 \\ & s_2 \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} u_1 & u_3 \\ & u_2 \end{smallmatrix}\right)$, then*

1. *For $i \in \{1, 2\}$, we have s_i are semisimple, u_i are unipotent and $g_i = s_i u_i = u_i s_i$ is the Jordan decomposition of g_i into semisimple and unipotent parts.*

(2) *s comes from $F_n^\times$ if and only if s_1 and s_2 come from $F_k^\times$ and $F_{n-k}^\times$ respectively, and have the same minimal polynomial of degree m dividing $gcd\{k,n\}$.*

Proof. (1) Let m be a positive integer such that $u^m = I_n$. Since $u^m = \begin{pmatrix} u_1^m & * \\ & u_2^m \end{pmatrix}$, it follows that u_1 and u_2 are unipotent. Since submodules and quotient modules of semisimple modules are themselves semisimple, it follows that s_1 and s_2 are semisimple. Also the condition $su = us = g$ gives $s_iu_i = u_is_i = g_i$ for $i = 1, 2$. This shows that $g_i = s_iu_i$ is the Jordan decomposition of g_i into their semisimple and unipotent parts.

(2) The characteristic polynomial of s which is $\det \begin{pmatrix} (XI_k - s_1) & -s_3 \\ & (XI_{n-k} - s_2) \end{pmatrix}$ equals $\det(XI_k - s_1) \cdot \det(XI_{n-k} - s_2)$. By definition s comes from $F_n^\times$ if and only if its characteristic polynomial $\det(XI_n - s) = \det(XI_k - s_1) \cdot \det(XI_{n-k} - s_2)$ is a power of an irreducible polynomial $P(X)$ of degree $m|n$. This is possible if and only if $\det(XI_{n-k} - s_2) = P(X)^{(n-k)/m}$ and $\det(XI_k - s_1) = P(X)^{k/m}$, or equivalently s_1 and s_2 come from $F_k^\times$ and $F_{n-k}^\times$ and have the same minimal polynomial. Clearly m divides $k, n-k$ and hence $\gcd\{k,n\}$. □

By definition 4.3 of $\Pi_{k,n}$, its character at $g = \begin{pmatrix} g_1 & g_1g_3 \\ & g_2 \end{pmatrix} \in P_{k,n}$ is

$$\begin{aligned} \Theta_{k,n}(g) &= \sum_{r \geq 1} \chi_{\mathrm{Ind}_{P_{k,n}(r)}^{P_{k,n}}(\psi_r \otimes [\mathrm{St}_{G_r} \otimes \Pi_{r,n-k}])}(g) \\ &= \sum_{r \geq 1} \tfrac{1}{|P_{k,n}(r)|} \sum_{t \in P_{k,n}} \chi_{\psi_r \otimes [\mathrm{St}_{G_r} \otimes \Pi_{r,n-k}]}(t^{-1}gt), \end{aligned} \tag{4.5}$$

where the inner sum $\sum_{t \in P_{k,n}}$ runs over those t such that $t^{-1}gt \in P_{k,n}(r)$.

Proposition 4.11. *For $g = \begin{pmatrix} g_1 & g_1g_3 \\ 0 & g_2 \end{pmatrix}$ for which the semisimple part of $g_1 \in G_k$ comes from $F_k^\times$, the character $\Theta_{k,n}$ of $\Pi_{k,n}$ at g is 0 if the semisimple part of g does not come from $F_n^\times$.*

Proof. We prove the assertion by induction on the 'length' $(n-k)$. If $n-k = 0$, the assertion is vacuously true, because $g = g_1$. For $n - k > 0$, we assume inductively that the assertion is true for $\Pi_{r,n-k}$ which has 'length' $(n-k-r) < n-k$. Let s_1, s_2 and s denote the semisimple parts of g_1, g_2 and g. Therefore, by Lemma 4.10, s does not come from $F_n^\times$ if and only if: either s_2 does not come from $F_{n-k}^\times$ or s_2 comes from $F_{n-k}^\times$ but with a minimal polynomial different from $p(X)$. In (4.5), if $t = \begin{pmatrix} t_1 & t_1t_3 \\ & t_2 \end{pmatrix}$ then $t^{-1}gt$ is of the form $\begin{pmatrix} t_1^{-1}g_1t_1 & * \\ & t_2^{-1}g_2t_2 \end{pmatrix}$. It is in $P_{k,n}(r)$ and hence is of the form $\begin{pmatrix} h_1 & & & \\ h_5 & h_4 & * & \\ & & h_1 & h_3 \\ & & & h_2 \end{pmatrix}$. If there is no $t \in P_{k,n}$ such that $t^{-1}gt$ is in $P_{k,n}(r)$, then clearly $\Theta_{k,n}(g) = 0$ by (4.5). So we now assume there is such a t. We are given that the semisimple part s_1 of g_1 comes from $F_k^\times$ and has minimal polynomial $p(X)$. Thus the semisimple part of $t_1^{-1}g_1t_1 = \begin{pmatrix} h_1 & \\ h_5 & h_4 \end{pmatrix}$ also comes from $F_k^\times$ and has minimal polynomial $p(X)$. As in Lemma 4.10, this is equivalent to the semisimple part of h_1, h_4 coming from $F_{\gcd\{r,k\}}^\times$. In particular, the semisimple part σ_1 of h_1 comes from $F_r^\times$ and has minimal polynomial $p(X)$. Turning now to $t_2^{-1}g_2t_2 = \begin{pmatrix} h_1 & h_3 \\ & h_2 \end{pmatrix}$, and using the fact that the semisimple part σ_1 of h_1 comes from

$F_r^\times$ with minimal polynomial $p(X)$, it follows that in case s_2 comes from $F_{n-k}^\times$, its minimal polynomial must be the same as that of s_1, namely $p(X)$. Therefore, by Lemma 4.10, that the condition g does not come from $F_n^\times$ is equivalent to the semisimple part s_2 of g_2 not coming from $F_{n-k}^\times$. So the semisimple part of $\left(\begin{smallmatrix} h_1 & h_3 \\ & h_2 \end{smallmatrix}\right)$ does not come from $F_{n-k}^\times$ but the semisimple part of h_1 does come from $F_r^\times$. Since we have assumed the inductive hypothesis for $\Pi_{r,n-k}$, we conclude that $\Theta_{r,n-k}(t_2^{-1}g_2t_2) = 0$. This completes the proof that $\Theta_{k,n}(g) = 0$ if s does not come from $F_n^\times$. $\square$

Step 2: Representatives $g_{\alpha,\beta,h}$ of the left cosets of $P_{k,n}(r)$ in $P_{k,n}$

As shown in Lemma 4.7, the index of $P_{k,n}(r)$ in $P_{k,n}$ is

$$\tfrac{|P_{k,n}|}{|P_{k,n}(r)|} = \tfrac{|G_k \times G_{n-k}|}{|P^*_{r,k} \times P_{r,n-k}|}\tfrac{|P^*_{r,k} \times P_{r,n-k}|}{|M_r|} = \left[\begin{smallmatrix} k \\ r \end{smallmatrix}\right]_q \left[\begin{smallmatrix} n-k \\ r \end{smallmatrix}\right]_q |G_r| = a(k \times n-k, r, q).$$

Since

$$M_r = \left\{ \begin{pmatrix} h_1 & & & \\ h_5 & h_4 & & \\ & & h_6 & h_3 \\ & & & h_2 \end{pmatrix} \in P^*_{r,k} \times P_{r,n-k} \colon h_1 = h_6 \right\},$$

it is clear that the set

$$\left\{ \begin{pmatrix} h & & \\ & I_{k-r} & \\ & & I_{n-k} \end{pmatrix} \in P^*_{r,k} \times P_{r,n-k} \colon h \in G_r \right\},$$

is a complete set of representatives of the left cosets of M_r in $P^*_{r,k} \times P_{r,n-k}$.

For each $(k-r)$ dimensional subspace α of F^k, let $g_\alpha \in G_k$ be a matrix whose last $(k-r)$ columns span α. For each r-dimensional subspace β of F^{n-k}, let $g_\beta \in G_{n-k}$ be a matrix whose first r columns span β. We claim that the collection of $\left[\begin{smallmatrix} k \\ k-r \end{smallmatrix}\right]_q \left[\begin{smallmatrix} n-k \\ r \end{smallmatrix}\right]_q$ matrices in $G_k \times G_{n-k} \subset P_{k,n}$ of the form $g_{\alpha,\beta} = \left(\begin{smallmatrix} g_\alpha & \\ & g_\beta \end{smallmatrix}\right)$ forms a complete set of representatives of the $\left[\begin{smallmatrix} k \\ k-r \end{smallmatrix}\right]_q \left[\begin{smallmatrix} n-k \\ r \end{smallmatrix}\right]_q$ left cosets of $P^*_{r,k} \times P_{r,n-k}$ in $G_k \times G_{n-k}$. It suffices to show that the cosets $g_{\alpha,\beta}P^*_{r,k} \times P_{r,n-k}$ are distinct. However, this is immediate from the fact that the coset $g_{\alpha,\beta}P^*_{r,k} \times P_{r,n-k}$ consists of those elements $\left(\begin{smallmatrix} g_1 & \\ & g_2 \end{smallmatrix}\right)$ of $G_k \times G_{n-k}$ for which the span of the last $k-r$ columns of g_1 is α, and the span of the first r columns of g_2 is β. In conclusion, the matrices $g_{\alpha,\beta,h} = \left(\begin{smallmatrix} g_\alpha \left(\begin{smallmatrix} h & \\ & I_{k-r} \end{smallmatrix}\right) & \\ & g_\beta \end{smallmatrix}\right)$ form a complete set of representatives for the left cosets of $P_{k,n}(r)$ in $P_{k,n}$. The matrix

$$g^{-1}_{\alpha,\beta,h}\, g\, g_{\alpha,\beta,h} = \begin{pmatrix} \begin{pmatrix} h^{-1} & \\ & I_{k-r} \end{pmatrix} g_\alpha^{-1} g_1 g_\alpha \begin{pmatrix} h & \\ & I_{k-r} \end{pmatrix} & \\ & g_\beta^{-1} g_2 g_\beta \end{pmatrix} \cdot \begin{pmatrix} I_k & \begin{pmatrix} h^{-1} & \\ & I_{k-r} \end{pmatrix} g_\alpha^{-1} g_3 g_\beta \\ & I_{n-k} \end{pmatrix}$$

is in $P_{k,n}(r)$ if and only if

$$\begin{pmatrix} \begin{pmatrix} h^{-1} & \\ & I_{k-r} \end{pmatrix} g_\alpha^{-1} g_1 g_\alpha \begin{pmatrix} h & \\ & I_{k-r} \end{pmatrix} & \\ & g_\beta^{-1} g_2 g_\beta \end{pmatrix} \text{ has the form } \begin{pmatrix} h_1 & & & \\ h_5 & h_4 & & \\ & & h_1 & h_3 \\ & & & h_2 \end{pmatrix} \in M_r.$$

We note that $h_4 \in G_{k-r}$ represents $g_1|_\alpha$ and h_1 represents the induced operator $\bar{g_1} : F^k/\alpha \to F^k/\alpha$ as well as the operator $g_2|_\beta : \beta \to \beta$. We now rewrite the inner sum (for fixed value of r in the outer sum) in the formula (4.5) in terms of the coset representatives $g_{\alpha,\beta,h}$:

$$\sum_{g_{\alpha,\beta,h}} \chi_{\psi_r \otimes [\mathrm{St}_{G_r} \otimes \Pi_{r,n-k}]}(g_{\alpha,\beta,h}^{-1}\, g\, g_{\alpha,\beta,h}), \tag{4.6}$$

where $\sum_{g_{\alpha,\beta,h}}$ runs over those $g_{\alpha,\beta,h}$ for which $g_{\alpha,\beta,h}^{-1}\, g\, g_{\alpha,\beta,h}$ lies in $P_{k,n}(r)$.

Step 3: The number of choices of $g_{\alpha,\beta,h}$ that contribute to (4.6)

Since $\mathrm{St}_{G_r}(h_1) = 0$ for non-semisimple h_1, we insist the operators $\bar{g_1} : F^k/\alpha \to F^k/\alpha$ and $g_2|_\beta : \beta \to \beta$ are semisimple, for otherwise $\mathrm{St}_{G_r}(h_1) = 0$ (in (4.6)) for the matrix h_1 that represents these operators. Since the case when s does not come from $F_n^\times$ has been addressed in Proposition 4.11, we now assume s comes from $F_n^\times$. By part (2) of Lemma 4.10, we must have that s_2 comes from $F_{n-k}^\times$ and its minimal polynomial is again $p(X)$. Therefore the semisimple part of $g_\beta^{-1} g_2 g_\beta = \left(\begin{smallmatrix} h_1 & h_3 \\ & h_2 \end{smallmatrix}\right)$ also comes from $F_{n-k}^\times$. Again by part (2) of Lemma 4.10, the semisimple part of h_1 comes from $F_r^\times$ and has minimal polynomial $p(X)$. Since h_1 is semisimple, we conclude that h_1 is *semisimple and comes from* $F_r^\times$. Since h_1 represents the operators $\bar{g_1} : F^k/\alpha \to F^k/\alpha$ and $g_2|_\beta : \beta \to \beta$, both these operators are semisimple and come from $F_r^\times$.

We consider F^k, F^{n-k} and F^n as $F[X]$-modules with X acting as the F-linear endomorphism g_1, g_2 and g respectively. Since the minimal polynomial of s_1, s_2 and s is $p(X)$, the characteristic polynomials of s_1, s_2 and s (and hence the characteristic polynomials of g_1, g_2 and g) are $p(X)^{k'}$, $p(X)^{n'-k'}$ and $p(X)^{n'}$ respectively. In particular we see F^k, F^{n-k} and F^n are $p(X)$-torsion modules. Thus we can write

$$F^k \simeq_{F[X]} \bigoplus_{i=1}^{t(g_1)} {}^{F[X]}\!/\!{}_{p(X)^{\mu_i}},\ F^n \simeq_{F[X]} \bigoplus_{i=1}^{t(g)} {}^{F[X]}\!/\!{}_{p(X)^{\epsilon_i}},\ F^{n-k} \simeq_{F[X]} \bigoplus_{i=1}^{t(g_2)} {}^{F[X]}\!/\!{}_{p(X)^{\nu_i}}$$

for some positive integers ν_i, μ_i and ϵ_i. We note that $(X-\lambda)$-torsion submodule of $F^k \otimes F_m$ is $\oplus_{i=1}^{t(g_1)} F_m[X]/(X-\lambda)^{\mu_i}$ and hence $t(g_1) = \dim\ker(g_1 - \lambda I_k)$ where the dimension is over any extension field of F_m, for example F_n. Similarly

$$t(g_1) = \dim_{F_n}\ker(g_1 - \lambda I_k),\ t(g) = \dim_{F_n}\ker(g - \lambda I_n),\ t(g_2) = \dim_{F_n}\ker(g_2 - \lambda I_{n-k}).$$

Consider the multiplication by $p(X)$-maps on each of the three $F[X]$-modules F^k, F^n and F^{n-k}. These fit into the following commutative diagram of $F[X]$-modules where the rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & K(g_1) \arrow[r] \arrow[d] & K(g) \arrow[r] \arrow[d] & K(g_2) \arrow[d] & \\
0 \arrow[r] & F^k \arrow[r] \arrow[d, "p(X)"] & F^n \arrow[r] \arrow[d, "p(X)"] & F^{n-k} \arrow[r] \arrow[d, "p(X)"] & 0 \\
0 \arrow[r] & F^k \arrow[r] & F^n \arrow[r] & F^{n-k} \arrow[r] & 0
\end{tikzcd} \tag{4.7}$$

and the kernel of the multiplication by $p(X)$-maps are:

$$
\begin{aligned}
K(g_1) &= \oplus_{i=1}^{t(g_1)} {}^{p(X)^{\mu_i-1}F[X]}\!/\!{}_{p(X)^{\mu_i}} \simeq \oplus^{t(g_1)}\, {}^{F[X]}\!/\!{}_{p(X)} \\
K(g) &= \oplus_{i=1}^{t(g)} {}^{p(X)^{\epsilon_i-1}F[X]}\!/\!{}_{p(X)^{\epsilon_i}} \simeq \oplus^{t(g)}\, {}^{F[X]}\!/\!{}_{p(X)}, \\
K(g_2) &= \oplus_{i=1}^{t(g_2)} {}^{p(X)^{\nu_i-1}F[X]}\!/\!{}_{p(X)^{\nu_i}} \simeq \oplus^{t(g_2)}\, {}^{F[X]}\!/\!{}_{p(X)}
\end{aligned}
$$

We also note the structure of $\operatorname{coker}(F^k \xrightarrow{p(X)} F^k)$: since $p(X)F^k = \oplus_{i=1}^{t(g_1)} {}^{p(X)F[X]}\!/\!{}_{p(X)^{\mu_i}}$, we see that

$$
\operatorname{coker}(F^k \xrightarrow{p(X)} F^k) \simeq_{F[X]} \oplus^{t(g_1)}\, {}^{F[X]}\!/\!{}_{p(X)}.
$$

For the $F[X]$-submodule α of F^k, we note that $\det(XI_k - g_1) = p(X)^{k'}$ is the product of the characteristic polynomials of $g_1|_\alpha$ and $\bar{g_1} : F^k/\alpha \to F^k/\alpha$. In particular, F^k/α is $p(X)$-torsion. Since F^k/α is semisimple, it follows that $F^k/\alpha \simeq_{F[X]} \oplus^{r'} F[X]/p(X)$, and hence $p(X)(F^k/\alpha) = 0$. This shows that $p(X)F^k \subset \alpha$, and hence $\bar{\alpha} := {}^{\alpha}\!/\!{}_{p(X)F^k}$ is a submodule of $\operatorname{coker}(F^k \xrightarrow{p(X)} F^k)$. Therefore,

$$
\oplus^{r'}\, {}^{F[X]}\!/\!{}_{p(X)} \simeq {}^{F^k}\!/\!{}_{\alpha} \simeq \operatorname{coker}(F^k \xrightarrow{p(X)} F^k)/\bar{\alpha} \simeq (\oplus^{t(g_1)}\, {}^{F[X]}\!/\!{}_{p(X)})/\bar{\alpha}.
$$

We note that the $F[X]$-module $\oplus^\ell F[X]/p(X)$ is annihilated by $p(X)$, and hence it naturally carries an $F[X]/p(X)$-module structure. Identifying $F[X]/p(X)$ with the field F_m, we identify the $F[X]$-module $\oplus^\ell F[X]/p(X)$ with the F_m-vector space F_m^ℓ. In this way

$$
F^k/\alpha \simeq F_m^{r'} \simeq F_m^{t(g_1)}/\bar{\alpha}.
$$

This shows that the number of ways to choose α is same as the number $\begin{bmatrix} t(g_1) \\ t(g_1)-r' \end{bmatrix}_{q'}$ of $(t(g_1)-r')$-dimensional F_m-linear subspaces of $F_m^{t(g_1)}$.

Similarly, the characteristic polynomial of $g_2|_\beta$ is $p(X)^{r'}$, and since $g_2|_\beta$ is semisimple, we see that $\beta \simeq \oplus^{r'} F[X]/p(X)$. Clearly $p(X)\beta = 0$ and hence β is a submodule of $K(g_2) \simeq \oplus^{t(g_2)} F[X]/p(X)$. As above, we may regard β as r' dimensional F_m-linear subspace of $F_m^{t(g_2)}$, and hence the number of ways of choosing β is $\begin{bmatrix} t(g_2) \\ r' \end{bmatrix}_{q'}$.

Let h_1 be the $r \times r$ submatrix on the first r rows and columns of $g_\beta^{-1} g_2 g_\beta$ as well as $\begin{pmatrix} h^{-1} & \\ & I_{k-r} \end{pmatrix} g_\alpha^{-1} g_1 g_\alpha \left(\begin{smallmatrix} h & \\ & I_{k-r} \end{smallmatrix}\right)$ which has the form $\left(\begin{smallmatrix} h_1 & \\ * & * \end{smallmatrix}\right)$. If h_1' is the $r \times r$ submatrix on the first r rows and columns of $g_\alpha^{-1} g_1 g_\alpha$, then we need $h^{-1} h_1' h = h_1$. Since h_1' represents $\bar{g_1} : F^k/\alpha \to F^k/\alpha$ and h_1 represents $g_2|_\beta : \beta \to \beta$, it follows that h is an invertible element of $\operatorname{Hom}_{F[X]}(\beta, F^k/\alpha)$. Since $\beta \simeq_{F[X]} F^k/\alpha \simeq_{F[X]} \oplus^{r'} F[X]/p(X)$, we can identify $\operatorname{Hom}_{F[X]}(\beta, F^k/\alpha)$ with $\operatorname{End}_{F_m}(F_m^{r'})$ and invertible elements of $\operatorname{Hom}_{F[X]}(\beta, F^k/\alpha)$ with $\mathrm{GL}_{r'}(F_m)$. Thus, the number of choices for h is $\#\,\mathrm{GL}_{r'}(F_m)$.

We now summarize the above discussion about the number of ways to pick α, β and h: Of the $a(k \times n-k, r, q)$ cosets $g_{\alpha,\beta,h}$ of $P_{k,n}(r)$ in $P_{k,n}$, for g as above, there are only

$$
a(t(g_1) \times t(g_2), r', q') = \begin{bmatrix} t(g_1) \\ r' \end{bmatrix}_{q'} \cdot \begin{bmatrix} t(g_2) \\ r' \end{bmatrix}_{q'} \cdot |\,\mathrm{GL}_{r'}(F_m)|
$$

cosets that contribute to (4.6).

For use in the next step, we count the number of choices $\begin{bmatrix} t(g_1) \\ r' \end{bmatrix}_{q'} \cdot \begin{bmatrix} t(g_2) \\ r' \end{bmatrix}_{q'}$ of the pair (α, β) in a finer way: Let $\mathrm{coker}(p(X))$ stand for $\mathrm{coker}(F^k \xrightarrow{p(X)} F^k)$. From the 'snake-lemma' applied to the diagram (4.7), there is a $F[X]$-module connecting homomorphism $K(g_2) \xrightarrow{C(g)} \mathrm{coker}(p(X))$, such that

$$K(g)/K(g_1) \simeq_{F[X]} \ker(C(g)).$$

Since β is a submodule of $K(g_2)$ and $F^k/\alpha \simeq \mathrm{coker}(p(X))/\bar{\alpha}$ as above, we define a map $C(g)_{\alpha,\beta}$ by the diagram below:

$$\begin{tikzcd}
K(g_2) \arrow[r, "C(g)"] & \mathrm{coker}(p(X)) \arrow[d, two heads] \\
\beta \arrow[u, hook] \arrow[r, "C(g)_{\alpha,\beta}"'] & F^k/\alpha
\end{tikzcd}$$

Since $K(g_2) \simeq_{F[X]} \oplus^{t(g_2)} F[X]/p(X)$ and $\mathrm{coker}(p(X)) \simeq_{F[X]} \oplus^{t(g_1)} F[X]/p(X)$, the map $C(g)$ is an element of $\mathrm{Hom}_{F[X]}(\oplus^{t(g_2)} F[X]/p(X), \oplus^{t(g_1)} F[X]/p(X))$ which we identify with $\mathrm{Hom}_{F_m}(F_m^{t(g_2)}, F_m^{t(g_1)})$. Let $t(g_3) := \dim_{F_m} \ker(C(g))$. From the fact that $K(g)/K(g_1) \simeq_{F[X]} \ker(C(g))$, we get

$$t(g) - t(g_1) = t(g_3).$$

We partition the number of choices $\begin{bmatrix} t(g_1) \\ r' \end{bmatrix}_{q'} \cdot \begin{bmatrix} t(g_2) \\ r' \end{bmatrix}_{q'}$ of the pairs (α, β) as

$$\begin{bmatrix} t(g_1) \\ r' \end{bmatrix}_{q'} \cdot \begin{bmatrix} t(g_2) \\ r' \end{bmatrix}_{q'} = \sum_s n_r(s),$$

where $n_r(s)$ is the number of pairs (α, β) such that the map

$$C(g)_{\alpha,\beta} \in \mathrm{Hom}_{F[X]}(\beta, F^k/\alpha) \simeq \mathrm{End}_{F_m}(F_m^{r'}),$$

has rank s. We now calculate $n_r(s)$ in terms of $t(g_3)$. Let $\dim_{F_m}(\beta \cap \ker C(g)) = r' - t$ for some integer t. The number of such $\beta \subset K(g_2)$ is

$$\begin{bmatrix} t(g_3) \\ r'-t \end{bmatrix}_{q'} \begin{bmatrix} t(g_2)-t(g_3) \\ t \end{bmatrix}_{q'} (q')^{t(t(g_3)-r'+t)}.$$

For such a β, the map $C(g)_{\alpha,\beta}$ has rank s if and only if $\dim_{F_m}(\bar{\alpha} \cap C(g)(\beta)) = t - s$. The number of such $\bar{\alpha} \subset \mathrm{coker}(p(X))$ is

$$\begin{bmatrix} t \\ t-s \end{bmatrix}_{q'} \begin{bmatrix} t(g_1)-t \\ t(g_1)-r'-t+s \end{bmatrix}_{q'} (q')^{(t(g_1)-r'-t+s)s}.$$

Therefore,

$$n_r(s) = \sum_t \left(\begin{bmatrix} t(g_3) \\ r'-t \end{bmatrix}_{q'} \begin{bmatrix} t(g_2)-t(g_3) \\ t \end{bmatrix}_{q'} (q')^{t(t(g_3)-r'+t)} \right) \left(\begin{bmatrix} t \\ t-s \end{bmatrix}_{q'} \begin{bmatrix} t(g_1)-t \\ t(g_1)-r'-t+s \end{bmatrix}_{q'} (q')^{(t(g_1)-r'-t+s)s} \right). \tag{4.8}$$

Step 4: The sum over h in (4.6) for fixed α, β

In the formula (4.6), we first fix a pair (α, β) such that the $F[X]$-submodule β of F^{n-k} and the quotient $F[X]$-module F^k/α are isomorphic to $\oplus^{r'} F[X]/p(X)$, and we calculate the sum

$$\sum_h \chi_{\psi_r \otimes [\mathrm{St}_{G_r} \otimes \Pi_{r,n-k}]}(g_{\alpha,\beta,h}^{-1}\, g\, g_{\alpha,\beta,h})$$

where h runs over the $F[X]$-module isomorphisms from β to F^k/α. The above sum can be written as

$$\sum_h \mathrm{St}_{G_r}(h_1) \cdot \Theta_{r,n-k}(g_\beta^{-1} g_2 g_\beta) \cdot \psi_0(\mathrm{Tr}(h^{-1} g_3(\alpha,\beta)),$$

where h_1 as before, is the $r \times r$ submatrix of $g_\beta^{-1} g_2 g_\beta$ on the first r rows and columns, and where $g_3(\alpha,\beta)$ is the $r \times r$ submatrix of $g_\alpha^{-1} g_3 g_\beta$ on the first r rows and columns. We note that $\mathrm{St}_{G_r}(h_1) = (-1)^{r-r'} q'^{\binom{r'}{2}}$ by Lemma 2.4 and the fact that h_1 is semsimple and comes from $F_r^\times$. Next we determine $\Theta_{r,n-k}(g_\beta^{-1} g_2 g_\beta)$. The semisimple part of $g_\beta^{-1} g_2 g_\beta$ is $g_\beta^{-1} s_2 g_\beta$ which comes from $F_{n-k}^\times$ because s_2 comes from $F_{n-k}^\times$. Also $g_\beta^{-1} g_2 g_\beta = \left(\begin{smallmatrix} h_1 & h_3 \\ & h_2 \end{smallmatrix}\right)$, with the semisimple part of h_1 (which is h_1 itself) coming from $F_r^\times$. Therefore by the inductive hypothesis, we have

$$\Theta_{r,n-k}(g_\beta^{-1} g_2 g_\beta) = (-1)^{n-k-r}((q')^{r'}; q')_{t(g_2)-r'}.$$

Therefore, the above sum can be written as:

$$(-1)^{n-k-r'}(q')^{\binom{r'}{2}}((q')^{r'}; q')_{t(g_2)-r'} \sum_h \psi_0(\mathrm{Tr}(h^{-1} g_3(\alpha,\beta)).$$

We will show that the term $\sum_h \psi_0(\mathrm{Tr}(h^{-1} g_3(\alpha,\beta))$ equals $(-1)^s (q')^{\binom{s}{2}+s(r'-s)} |\operatorname{GL}_{r'-s}(F_m)|$ where s is the rank of $C(g)_{\alpha,\beta}$ viewed as an element of $\mathrm{End}_{F_m}(F_m^{r'})$ (as in the previous step). Therefore, the above sum becomes

$$(-1)^{n-k-r'}(q')^{\binom{r'}{2}}((q')^{r'}; q')_{t(g_2)-r'}(-1)^s (q')^{\binom{s}{2}+s(r'-s)} |\operatorname{GL}_{r'-s}(F_m)| n_r(s) \quad (4.9)$$

where $n_r(s)$ is determined in (4.8). We note that $g_3(\alpha,\beta)$ represents the induced map

$$\beta \xrightarrow{g_3|_\beta} F^k/\alpha.$$

We recall that

$$K(g_2) \simeq_{F[X]} \oplus^{t(g_2)} \tfrac{F[X]}{p(X)}, \quad \operatorname{coker}(p(X)) = \tfrac{F^k}{p(X)F^k} \simeq_{F[X]} \oplus^{t(g_1)} \tfrac{F[X]}{p(X)}.$$

It will be useful to choose generators $v_1, \ldots, v_{t(g_2)}$ and $u_1, \ldots, u_{t(g_1)}$ such that

$$K(g_2) \simeq_{F[X]} \bigoplus_{j=1}^{t(g_2)} \tfrac{F[X]}{p(X)} v_j, \qquad \tfrac{F^k}{p(X)F^k} \simeq_{F[X]} \bigoplus_{i=1}^{t(g_1)} \tfrac{F[X]}{p(X)} u_i, \quad (4.10)$$

$$\beta \simeq_{F[X]} \bigoplus_{j=1}^{r'} \tfrac{F[X]}{p(X)} v_j, \quad \bar{\alpha} \simeq_{F[X]} \bigoplus_{i=r'+1}^{t(g_1)} \tfrac{F[X]}{p(X)} u_i, \quad F^k/\alpha \simeq_{F[X]} \bigoplus_{i=1}^{r'} \tfrac{F[X]}{p(X)} u_i.$$

By extending the scalars from F to F_m, the $F_m[X]$-modules $\beta \otimes F_m \subset K(g_2) \otimes_F F_m$, and $F_m^k/(\alpha \otimes F_m) \subset F_m^k/p(X)F_m^k$ decompose as a direct sum of eigenspaces for the eigenvalues $\phi^\ell(\lambda), 0 \leq \ell \leq m-1$, which we denote as $\beta(\phi^\ell(\lambda)) \subset K(g_2, \phi^\ell(\lambda))$ and $\alpha'(\phi^\ell(\lambda)) \subset U(\phi^\ell(\lambda))$ respectively:

$$K(g_2) \otimes_F F_m \simeq_{F_m[X]} \oplus_{\ell=0}^{m-1} K(g_2, \phi^\ell(\lambda)), \quad K(g_2, \lambda) = \oplus_{j=1}^{t(g_2)} \tfrac{F[X]}{X-\lambda} v_j,$$

$$\tfrac{F_m^k}{p(X)F_m^k} \simeq_{F_m[X]} \oplus_{\ell=0}^{m-1} U(\phi^\ell(\lambda)), \quad U(\lambda) = \oplus_{i=1}^{t(g_1)} \tfrac{F[X]}{X-\lambda} u_i,$$

$$\beta \otimes_F F_m \simeq_{F_m[X]} \oplus_{\ell=0}^{m-1} \beta(\phi^\ell(\lambda)), \quad \beta(\lambda) = \oplus_{j=1}^{r'} \tfrac{F[X]}{X-\lambda} v_j,$$

$$\tfrac{F_m^k}{\alpha \otimes F_m} \simeq_{F_m[X]} \oplus_{\ell=0}^{m-1} \alpha'(\phi^\ell(\lambda)), \quad \alpha'(\lambda) = \oplus_{i=1}^{r'} \tfrac{F[X]}{X-\lambda} u_i$$

Here, $K(g_2, \phi^\ell(\lambda)) = \phi^\ell(K(g_2, \lambda))$ and similarly for $U(\phi^\ell(\lambda))$, $\beta(\phi^\ell(\lambda))$ and $\alpha'(\phi^\ell(\lambda))$.

Viewing the connecting homomorphism $C(g) \in \mathrm{Hom}_{F[X]}(K(g_2), \mathrm{coker}(p(X)))$ as an element of $\mathrm{Hom}_{F_m[X]}(K(g_2) \otimes F_m, F_m^k/p(X)F_m^k)$, there is an element $C(g, \lambda) \in \mathrm{Hom}_{F_m[X]}(K(g_2, \lambda), U(\lambda))$ such that $C(g) = \oplus_{\ell=0}^{m-1} K(g_2, \phi^\ell(\lambda)) \xrightarrow{C(g,\phi^\ell(\lambda))} \oplus_{\ell=0}^{m-1} U(\phi^\ell(\lambda))$ with $C(g, \phi^\ell(\lambda)) = \phi^\ell \circ C(g, \lambda) \circ \phi^{-\ell}$. Similarly the map $C(g)_{\alpha,\beta} \in \mathrm{Hom}_{F[X]}(\beta, F^k/\alpha)$ when viewed as an element of $\mathrm{Hom}_{F_m[X]}(\beta \otimes F_m, F_m^k/(\alpha \otimes F_m))$ is described by a map $C(g, \lambda)_{\alpha,\beta} \in \mathrm{Hom}_{F_m[X]}(\beta(\lambda), \alpha'(\lambda))$. In terms of the generators $K(g_2, \lambda) = \bigoplus_{j=1}^{t(g_2)} \frac{F[X]}{X-\lambda} v_j$ and $U(\lambda) = \bigoplus_{i=1}^{t(g_1)} \frac{F[X]}{X-\lambda} u_i$, the map $C(g, \lambda)$ is described by a matrix

$$A(g, \lambda) \in \mathrm{M}_{t(g_2)\times t(g_1)}(\tfrac{F_m[X]}{X-\lambda}) \simeq \mathrm{M}_{t(g_2)\times t(g_1)}(F_m), \qquad C(g, \lambda)v_j = \sum_i A(g, \lambda)_{i,j} u_i.$$

The submatrix $A(g, \lambda)_{\alpha,\beta} \in \mathrm{M}_{r'\times r'}(F_m)$ of $A(g, \lambda)$ on the first r' rows and columns describes the map $C(g, \lambda)_{\alpha,\beta}$. The matrix $g_3 \in \mathrm{M}_{n-k\times k}(F)$, defines a linear map $F^{n-k} \xrightarrow{g_3} F^k$. Consider the map

$$g_3' \in \mathrm{Hom}_F(K(g_2), \mathrm{coker}(p(X))), \text{ defined by } K(g_2) \hookrightarrow F^{n-k} \xrightarrow{g_3} F^k \to \tfrac{F^k}{p(X)F^k}.$$

We also consider the linear map

$$g_3(\alpha, \beta) \in \mathrm{Hom}_F(\beta, F^k/\alpha), \text{ defined by } \beta \hookrightarrow K(g_2) \xrightarrow{g_3'} \tfrac{F^k}{p(X)F^k} \to \tfrac{F^k}{\alpha}.$$

Extending the scalars from F to F_m, we consider g_3' as an element of

$$g_3' \in \mathrm{Hom}_{F_m}(K(g_2) \otimes F_m, \tfrac{F_m^k}{p(X)F_m^k}) = \bigoplus_{j=0}^{m-1} \bigoplus_{i=0}^{m-1} \mathrm{Hom}_{F_m}(K(g_2, \phi^j(\lambda)), U(\phi^i(\lambda))).$$

There are maps $g_3'(\phi^i(\lambda), \phi^j(\lambda)) \in \mathrm{Hom}_{F_m}(K(g_2, \phi^j(\lambda)), U(\phi^i(\lambda)))$ such that

$$g_3'(\phi^i(\lambda), \phi^j(\lambda)) = \phi^j \circ g_3'(\phi^{i-j}(\lambda), \lambda) \circ \phi^{-j},$$

and hence is completely determined by $g_3'(\phi^\ell(\lambda), \lambda)$ for $\ell \in \{0, m-1\}$. Again in terms of the generators $K(g_2, \lambda) = \bigoplus_{j=1}^{t(g_2)} \frac{F[X]}{X-\lambda} v_j$ and $U(\lambda) = \bigoplus_{i=1}^{t(g_1)} \frac{F[X]}{X-\lambda} u_i$, the maps $g_3'(\phi^\ell(\lambda), \lambda)$ for $\ell \in \{0, m-1\}$ are described by matrices $B(g, \phi^\ell(\lambda))$ defined by $g_3'(\phi^\ell(\lambda), \lambda)(v_j) = \sum_i B(g, \phi^\ell(\lambda))u_i$. Similarly, the map $g_3'(\alpha, \beta) \in \mathrm{Hom}_{F_m}(\beta \otimes F_m, F_m^k/(\alpha \otimes F_m))$ is completely described by the associated maps $g_3'(\alpha, \beta)(\phi^\ell(\lambda), \lambda)$ for $\ell \in \{0, m-1\}$. The submatrix $B(g, \phi^\ell(\lambda))_{\alpha,\beta} \in \mathrm{M}_{r'\times r'}(F_m)$ of $B(g, \phi^\ell(\lambda))$ on the first r' rows and columns of $B(g, \phi^\ell(\lambda))$ describes the map $g_3'(\alpha, \beta)(\phi^\ell(\lambda), \lambda)$. The next lemma relates the matrices $A(g, \lambda)$ and $B(g, \lambda) \in \mathrm{M}_{t(g_2)\times t(g_1)}(F_m)$.

Lemma 4.12. *$A(g, \lambda) = c_\lambda B(g, \lambda) \in \mathrm{M}_{t(g_2)\times t(g_1)}(F_m)$ where $c_\lambda \in F_m^\times$ equals $\lambda \prod_{i=1}^{m-1}(\lambda - \phi^i(\lambda))$. Considering the submatrices on the first r' rows and columns of both matrices, we get*

$$A(g, \lambda)_{\alpha,\beta} = c_\lambda B(g, \lambda)_{\alpha,\beta} \in \mathrm{M}_{r'\times r'}(F_m).$$

Proof. The proof is obtained by chasing the diagram (4.7). We lift $v \in K(g_2, \lambda) \subset F_m^{n-k}$ to $(0, v) \in F_m^n$, and then apply $p(g)$ to $(0, v)$. Writing $p(X) = \prod_{i=1}^m (X - \phi^{m-i}(\lambda))$, we see $p(g) = \left(\begin{smallmatrix} p(g_1) & p(g)_3 \\ & p(g_2) \end{smallmatrix}\right)$ where

$$p(g)_3 = \sum_{i=0}^{m-1} (g_1 - \phi^{m-1}(\lambda)) \ldots (g_1 - \phi^{m-i}(\lambda))\, g_1 g_3\, (g_2 - \phi^{m-i-2}(\lambda)) \ldots (g_2 - \lambda).$$

Therefore, for $v \in K(g_2, \lambda)$, we have

$$p(g) \left(\begin{smallmatrix} 0 \\ v \end{smallmatrix} \right) = \begin{pmatrix} (g_1 - \phi^{m-1}(\lambda)) \dots (g_1 - \phi(\lambda)) g_1 g_3 v \\ 0 \end{pmatrix}.$$

We then have $C(g, \lambda)v \in U(\lambda)$ equals the projection of $(g_1 - \phi^{m-1}(\lambda)) \dots (g_1 - \phi(\lambda)) g_1 g_3 v \in F_m^k$ on $U(\lambda)$. Since g_1 acts as multiplication by λ on $U(\lambda)$, we have

$$C(g, \lambda)v = c_\lambda \, g_3 v, \quad c_\lambda = \lambda \prod_{i=1}^{m-1} (\lambda - \phi^i(\lambda)).$$

Since the matrix $A(g, \lambda)$ and $B(g, \lambda)$ represent the map $C(g, \lambda) \in \mathrm{Hom}_{F_m[X]}(K(g_2, \lambda), U(\lambda))$, and $g_3' \in \mathrm{Hom}_{F_m}(K(g_2, \lambda), U(\lambda))$ respectively, the proof is complete. □

For $h \in \mathrm{Hom}_{F[X]}(\beta, F^k/\alpha)$ invertible, the inverse map h^{-1} viewed as an element of $\mathrm{Hom}_{F_m[X]}(F_m^k/(\alpha \otimes F_m), \beta \otimes F_m)$ is of the form $\oplus_{i=0}^{m-1} \alpha'(\phi^i(\lambda)) \xrightarrow{h^{-1}_{\phi^i(\lambda)}} \oplus_{i=0}^{m-1} \beta(\phi^i(\lambda))$ with $h^{-1}_{\phi^i(\lambda)} = \phi^i \circ h_\lambda^{-1} \circ \phi^{-i}$. We now consider the trace of the composite map $\beta \xrightarrow{g_3(\alpha,\beta)} F^k/\alpha \xrightarrow{h^{-1}} \beta$. We have

$$\mathrm{Tr}(h^{-1} g_3(\alpha, \beta)) = \sum_{i=0}^{m-1} \mathrm{Tr}(h^{-1}_{\phi^i(\lambda)} g_3'(\alpha, \beta)(\phi^i(\lambda), \phi^i(\lambda))) = \mathrm{Tr}_{F_m/F} \mathrm{Tr}(h_\lambda^{-1} g_3'(\alpha, \beta)(\lambda, \lambda)).$$

The map $g_3'(\alpha, \beta)(\lambda, \lambda)$ is represented by the matrix $B(g, \lambda)_{\alpha,\beta}$ which equals $c_\lambda^{-1} A(g, \lambda)_{\alpha,\beta}$ by Lemma 4.12.

We recall that $A(g, \lambda)_{\alpha,\beta} \in \mathrm{M}_{r' \times r'}(F_m)$ represents the map $C(g, \lambda)_{\alpha,\beta}$. Let $h_0 \in \mathrm{GL}_{r'}(F_m)$ denote the matrix of h_λ^{-1} with respect to the generators $u_1, \dots, u_{r'}$ of $\alpha'(\lambda) = \oplus_{i=1}^{r'} \frac{F[X]}{X - \lambda} u_i$ and $v_1, \dots, v_{r'}$ of $\beta(\lambda) = \oplus_{j=1}^{r'} \frac{F[X]}{X - \lambda} v_j$. We now have

$$\mathrm{Tr}(h^{-1} g_3(\alpha, \beta)) = \mathrm{Tr}_{F_m/F} \mathrm{Tr}(c_\lambda^{-1} h_0 A(g, \lambda)_{\alpha,\beta}).$$

We note that $\tilde{\psi}_0 = \psi_0 \circ \mathrm{Tr}_{F_m/F}$ is a non-trivial additive character of F_m, and hence we have by Theorem 3.1:

$$\begin{aligned} \sum_{h \in \mathrm{Hom}_{F[X]}(\beta, F^k/\alpha)} \psi_0(\mathrm{Tr}(h^{-1} g_3(\alpha, \beta))) &= \sum_{h_0 \in \mathrm{GL}_{r'}(F_m)} \tilde{\psi}_0(\mathrm{Tr}\, c_\lambda^{-1} h_0^{-1} A(g, \lambda)_{\alpha,\beta}) \\ &= (-1)^s (q')^{\binom{s}{2} + s(r'-s)} |\, \mathrm{GL}_{r'-s}(F_m)|, \end{aligned}$$

where s is the rank of $C(g)_{\alpha,\beta}$.

Step 5: Simplification of (4.9)

Using the expression (4.8) for $n_r(s)$ in (4.9) and summing this over $r \geq 1$, we see that the character of $\sum_{r\geq 1} \operatorname{Ind}_{P_{k,n}(r)}^{P_{k,n}}(\psi_r \otimes [\mathrm{St}_{G_r} \otimes \Pi_{r,n-k}])$ at g is:

$$(-1)^{n-k}\left(\sum_{r\geq 1}(q')^{\binom{r'}{2}}((q')^r;q')_{t(g_2)-r'}\right)\cdot\left(\sum_t (-1)^{r'-t}\genfrac{[}{]}{0pt}{}{t(g_3)}{r'-t}_{q'}\genfrac{[}{]}{0pt}{}{t(g_2)-t(g_3)}{t}_{q'}(q')^{t(t(g_3)-r'+t)}\right)$$
$$\cdot\left(\sum_s (-1)^{t-s}(q')^{\binom{s}{2}+s(r'-s)}|\operatorname{GL}_{r'-s}(F_m)|\genfrac{[}{]}{0pt}{}{t}{t-s}_{q'}\genfrac{[}{]}{0pt}{}{t(g_1)-t}{t(g_1)-r'-t+s}_{q'}(q')^{(t(g_1)-r'-t+s)s}\right). \tag{4.11}$$

Let I_s denote the third parenthetical term in the above expression. We will show that I_s equals $\genfrac{[}{]}{0pt}{}{t(g_1)-t}{r'-t}_{q'}|\operatorname{GL}_{r'-t}(F_m)|(q')^{t(r'-t)+\binom{t}{2}}$. To do so, we first write

$$\genfrac{[}{]}{0pt}{}{t(g_1)-t}{t(g_1)-r'-t+s}_{q}|\operatorname{GL}_{r'-s}(F_m)| = (q')^{\binom{r'-s}{2}}\prod_{i=0}^{r'-s-1}((q')^{t(g_1)-t-i}-1),$$

using which the term I_s can be rewritten as

$$(q')^{\binom{r'}{2}}(\prod_{i=t}^{r'-1}((q')^{t(g_1)-i}-1))\sum_s(-1)^{t-s}\genfrac{[}{]}{0pt}{}{t}{t-s}_{q'}(q')^{s(t(g_1)-r'-t+s)}\prod_{i=0}^{t-s-1}((q')^{t(g_1)-r'-i}-1),$$

which in turn can be rewritten as:

$$(q')^{\binom{r'}{2}+t(t(g_1)-r')-\binom{t}{2}}(\prod_{i=t}^{r'-1}((q')^{t(g_1)-i}-1))\sum_s(-1)^{t-s}\genfrac{[}{]}{0pt}{}{t}{t-s}_{q'}(q')^{\binom{s}{2}}((q')^{r'-t(g_1)};q')_{t-s}.$$

The expression $\sum_s(-1)^{t-s}\genfrac{[}{]}{0pt}{}{t}{t-s}_{q'}(q')^{\binom{s}{2}}((q')^{r'-t(g_1)};q')_{t-s}$ above, equals $(q')^{\binom{t}{2}}(q')^{(r'-t(g_1))t}$ by Lemma 3.3. Therefore the term I_s equals

$$\begin{aligned}&(q')^{\binom{t}{2}+(r'-t(g_1))t}(q')^{\binom{r'}{2}+t(t(g_1)-r')-\binom{t}{2}}(\prod_{i=t}^{r'-1}((q')^{t(g_1)-i}-1))\\ &\qquad\qquad = (q')^{\binom{r'}{2}-\binom{r'-t}{2}}|\operatorname{GL}_{r'-t}(F_m)|\prod_{i=1}^{r'-t}(\tfrac{(q')^{t(g_1)-t-(i-1)}-1}{(q')^i-1}),\end{aligned}$$

which can be rewritten as

$$\genfrac{[}{]}{0pt}{}{t(g_1)-t}{r'-t}_{q'}|\operatorname{GL}_{r'-t}(F_m)|(q')^{t(r'-t)+\binom{t}{2}},$$

which proves the claim about the term I_s. The expression (4.11) can now be written as

$$\begin{aligned}&(-1)^{n-k}\left(\sum_{r'\geq 1}(q')^{\binom{r'}{2}}((q')^{r'};q')_{t(g_2)-r'}\right)\cdot\\ &\qquad\left(\sum_t(-1)^{r'-t}\genfrac{[}{]}{0pt}{}{t(g_3)}{r'-t}_{q'}\genfrac{[}{]}{0pt}{}{t(g_1)-t}{r'-t}_{q'}|\operatorname{GL}_{r'-t}(F_m)|\genfrac{[}{]}{0pt}{}{t(g_2)-t(g_3)}{t}_{q'}(q')^{\binom{t}{2}+t(g_3)t}\right).\end{aligned}$$

Writing

$$(q')^{\binom{r'}{2}}((q')^{r'};q')_{t(g_2)-r'} = (-1)^{t(g_3)-r'+t}(q')^{\binom{t(g_3)+t}{2}}((q')^{t(g_3)+t};q')_{t(g_2)-t(g_3)-t}((q')^{1-t(g_3)-t};q')_{t(g_3)-r'+t},$$

and using the fact that $\left[{t(g_3) \atop r'-t}\right]_{q'} \left[{t(g_1)-t \atop r'-t}\right]_{q'} |\operatorname{GL}_{r'-t}(F_m)| = a(t(g_1)-t\times t(g_3), r'-t, q')$ we can rewrite (4.11) as

$$(-1)^{n-k}\left(\sum_t (-1)^{t(g_3)}(q')^{\binom{t(g_3)+t}{2}+\binom{t}{2}+t(g_3)t}((q')^{t(g_3)+t};q')_{t(g_2)-t(g_3)-t}\left[{t(g_2)-t(g_3) \atop t}\right]_{q'}\right)$$

$$\cdot\left(\sum_{r'} a(t(g_1)-t\times t(g_3), r'-t, q')({q'}^{1-t(g_3)-t};q')_{t(g_3)-r'+t}\right).$$

By Lemma 3.4, the second term in the product above is

$$(q')^{t(g_3)(t(g_1)-t)}\,((q')^{1-t(g_3)-t(g_1)};q')_{t(g_3)} = (-1)^{t(g_3)}(q')^{-t(g_3)t-\binom{t(g_3)}{2}}((q')^{t(g_1)};q')_{t(g_3)}.$$

Therefore, we can rewrite (4.11) as

$$(-1)^{n-k}(q')^{\binom{t(g_2)}{2}-\binom{t(g_3)}{2}}((q')^{t(g_1)};q')_{t(g_3)}\cdot$$

$$\left(\sum_t (-1)^{t(g_2)-t(g_3)-t}\left[{t(g_2)-t(g_3) \atop t(g_2)-t(g_3)-t}\right]_{q'}((q')^{1-t(g_2)};q')_{t(g_2)-t(g_3)-t}(q')^{\binom{t}{2}}\right).$$

The last parenthetical term above equals

$$(q')^{(1-t(g_2))(t(g_2)-t(g_3))}(q')^{\binom{t(g_2)-t(g_3)}{2}} = (q')^{\binom{t(g_3)}{2}-\binom{t(g_2)}{2}},$$

by Lemma 3.3. Therefore (4.11) equals

$$(-1)^{n-k}((q')^{t(g_1)};q')_{t(g_3)} = (-1)^{n-k}((q')^{t(g_1)};q')_{t(g)-t(g_1)}$$

as was to be shown. This completes the proof of Theorem 4.9. □

4.4. Universal property of $\Pi_{k,n}$ with respect to cuspidal representations.

The representation $(\Phi^+)^{n-1}(1)$ of Mir_n satisfies a well known universal property with respect to restriction to Mir_n of cuspidal representations of G_n (for example [BZ76, 5.18]):

$$\pi_\theta|_{\operatorname{Mir}_n} = (\Phi^+)^{n-1}(1).$$

We can express this in terms of the representation $\Pi_{1,n}$. Using the representation $\Pi^\dagger_{1,n} = 1_{F^\times}\boxtimes(\Phi^+)^{n-1}(1)$ of $P_{n-1,n} = F^\times\times\operatorname{Mir}_n$ (from Lemma 4.6), we can rewrite this universal property as

$$\pi_\theta|_{P_{n-1,n}} = \Pi^\dagger_{1,n}\otimes\theta|_{F^\times},$$

where $\theta|_{F^\times}(g) = \theta(g_{n,n})$. Applying † to this, we get

$$\pi^\dagger_\theta|_{P_{1,n}} = \Pi_{1,n}\otimes\bar\theta|_{F^\times},$$

where $\bar\theta|_{F^\times}(g) = \bar\theta(g_{1,1})$. Using (4.4) in this equation, we get $\pi_{\bar\theta}|_{P_{1,n}} = \Pi_{1,n}\otimes\bar\theta|_{F^\times}$. Finally, replacing $\bar\theta$ with θ, we get

$$\pi_\theta|_{P_{1,n}} = \Pi_{1,n}\otimes\theta|_{F^\times}.$$

In summary, we have

$$\pi_\theta|_{P_{n-1,n}} = \Pi^\dagger_{1,n}\otimes\theta|_{F^\times}, \qquad \pi_\theta|_{P_{1,n}} = \Pi_{1,n}\otimes\theta|_{F^\times}. \tag{4.12}$$

Parts (2)-(3) of the next theorem generalize these properties (4.12) to $P_{k,n}$, and are consequences of part (1) of the theorem, which is much more general and applies not only to cuspidal representations but also to the class functions Θ_θ:

Theorem 4.13. *Let $P_{k,n}$ be a maximal parabolic subgroup of G_n. For $g = \left(\begin{smallmatrix} g_1 & g_1g_3 \\ 0 & g_2 \end{smallmatrix}\right) \in P_{k,n}$, let $p_1\colon P_{k,n} \to G_k$ be the epimorphism $p_1(g) = g_1$, and let $p_2\colon P_{k,n} \to G_{n-k}$ be the epimorphism $p_2(g) = g_2$. Let $d = gcd\{k, n\}$.*

1. *Let θ be any character of $F_n^\times$, and let θ_k be any character of $F_k^\times$ whose restriction to $F_d^\times$ is $\theta|_{F_d^\times}$ (such a θ_k exists by part* (1) *of Lemma 2.1). Let Θ_θ and Θ_{θ_k} be the associated class functions on G_n and G_k respectively (see Definition 2.2). We recall that $\Theta_{k,n}$ denotes the character of $\Pi_{k,n}$. For $g \in P_{k,n}$, we have:*
$$\Theta_\theta(g) = \Theta_{k,n}(g) \cdot \Theta_{\theta_k}(g_1).$$
2. *Let θ be a regular character of $F_n^\times$ and let π_θ be the associated cuspidal representation of G_n. If either $k \nmid n$, or if $k \mid n$ and $\theta|_{F_k^\times}$ is a regular character of $F_k^\times$, there is a cuspidal representation $\pi_{\tilde\theta}$ of G_k such that*
$$\pi_\theta|_{P_{k,n}} \simeq \Pi_{k,n} \otimes \pi_{\tilde\theta},$$
where the representation $\pi_{\tilde\theta}$ of G_k is inflated to $P_{k,n}$ via p_1. In the former case $k \nmid n$, we can take $\tilde\theta$ (by part (2) *of Lemma 2.1) to be a regular character of $F_k^\times$ such that $\tilde\theta|_{F_d^\times} = \theta|_{F_d^\times}$. In the latter case $\tilde\theta = \theta|_{F_k^\times}$.*
3. *Let π_θ be as in part (2) above. If either $(n-k) \nmid n$, or if $(n-k) \mid n$ and $\theta|_{F_{n-k}^\times}$ is a regular character of $F_{n-k}^\times$, there is a cuspidal representation $\pi_{\tilde\theta}$ of G_{n-k} such that*
$$\pi_\theta|_{P_{k,n}} \simeq \Pi^\dagger_{n-k,n} \otimes \pi_{\tilde\theta},$$
where the representation $\pi_{\tilde\theta}$ of G_{n-k} is inflated to $P_{k,n}$ via p_2. In the former case $(n-k) \nmid n$, we can take $\tilde\theta$ (by part (2) *of Lemma 2.1) to be a regular character of $F_{n-k}^\times$ such that $\tilde\theta|_{F_d^\times} = \theta|_{F_d^\times}$. In the latter case $\tilde\theta = \theta|_{F_{n-k}^\times}$.*

Proof. (1) Let $g = su$ be decomposition of $g = \left(\begin{smallmatrix} g_1 & g_1g_3 \\ & g_2 \end{smallmatrix}\right)$ into its semisimple and unipotent parts, with $s = \left(\begin{smallmatrix} s_1 & s_3 \\ & s_2 \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} u_1 & u_3 \\ & u_2 \end{smallmatrix}\right)$. By Lemma 4.10, s_1 and s_2 are the semisimple parts of g_1 and g_2 respectively. If s_1 does not come from $F_k^\times$, then s also does not come from $F_n^\times$, and hence $\Theta_\theta(g)$ and $\Theta_{\theta_k}(g_1)$ are both zero by Definition 2.2, which verifies $\Theta_\theta(g) = \Theta_{k,n}(g) \cdot \Theta_{\theta_k}(g_1)$. We now assume s_1 comes from $F_k^\times$. In this case if s does not come from $F_n^\times$, then $\Theta_{k,n}(g) = 0$ by Theorem 4.9, and $\Theta_\theta(g) = 0$ by Definition 2.2, which again verifies $\Theta_\theta(g) = \Theta_{k,n}(g) \cdot \Theta_{\theta_k}(g_1)$. We now assume s_1 comes from $F_k^\times$ and s comes from $F_n^\times$ (and hence both come from $F_d^\times$). If $\lambda \in F_m$ is a root of the minimal polynomial $p(X)$ of s, then by Theorem 4.9 and Definition 2.2 we have

$$\begin{aligned}\Theta_{k,n}(g) \cdot \Theta_{\theta_k}(g_1) &= \left[(-1)^{n-k}({q'}^{t(g_1)}; q')_{t(g)-t(g_1)}\right] \cdot \left[(-1)^{k-1}\left(\sum_{i=0}^{m-1} \theta(\phi^i(\lambda))\right)(q'; q')_{t(g_1)}\right] \\ &= (-1)^{n-1}\left(\sum_{i=0}^{m-1} \theta(\phi^i(\lambda))\right)(q'; q')_{t(g)} = \Theta_\theta(g).\end{aligned}$$

(2) This simply follows from part (1), by noting that Θ_{θ_k} is the character of the cuspidal representation $\pi_{\tilde\theta}$ of G_k.

(3) By part (2), there is a cuspidal representation $\pi_{\tilde{\theta}}$ of G_{n-k} such that $\pi_\theta|_{P_{n-k,n}} \simeq \Pi_{n-k,n} \otimes \pi_{\tilde{\theta}}$. From this we get

$$\pi_\theta^\dagger|_{P_{k,n}} \simeq \Pi_{n-k,n}^\dagger \otimes \pi_{\tilde{\theta}}^\dagger.$$

Using (4.4), we get

$$\pi_{\bar{\theta}}|_{P_{k,n}} \simeq \Pi_{n-k,n}^\dagger \otimes \pi_{\tilde{\tilde{\theta}}},$$

Finally, replacing the regular character $\bar{\theta}$ with θ, we get

$$\pi_\theta|_{P_{k,n}} \simeq \Pi_{n-k,n}^\dagger \otimes \pi_{\tilde{\theta}},$$

where the cuspidal representation $\pi_{\tilde{\theta}}$ of G_{n-k} is inflated to $P_{k,n}$ via p_2.

□

The inverse transpose automorphism $g \mapsto g^{-\top}$ carries the subgroup $P_{k,n}^*$ of G_n consisting of matrices of the form $g = \left(\begin{smallmatrix} g_1 & \\ g_5 g_1 & g_4 \end{smallmatrix}\right)$ to the parabolic subgroup $P_{k,n}$. This allows us to define:

Definition 4.14. We define a representation $\Pi_{k,n}^*$ of $P_{k,n}^*$ by

$$\Pi_{k,n}^*(g) = \Pi_{k,n}(g^{-\top}).$$

The character of $\Pi_{k,n}^*$ will be denoted $\Theta_{k,n}^*$. The results of Theorem 4.13 have obvious analogues for the representation $\Pi_{k,n}^*$ of $P_{k,n}^*$. We only record here the first property. Let θ and θ_k be as in part (1) of Theorem 4.13. For $g = \left(\begin{smallmatrix} g_1 & \\ g_5 g_1 & g_4 \end{smallmatrix}\right) \in P_{k,n}^*$, we have:

$$\Theta_\theta(g) = \Theta_{k,n}^*(g) \cdot \Theta_{\theta_k}(g_1). \tag{4.13}$$

5. STRUCTURE OF $\pi_{N,\psi}$

As before, let π_θ be a cuspidal representation of G_n, and $P = MN$ a maximal parabolic subgroup of G_n associated with the partition $(k, n-k)$ of n, and ψ a rank r character of the unipotent radical $N \simeq \mathrm{M}_{k\times n-k}(F)$. In this section we determine the structure of the twisted Jacquet module $\pi_{N,\psi}$ for general r. We recall from (2.2) that M_ψ (which we denote M_r here) equals:

$$M_r = \left\{ \left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right) : g_1 = \left(\begin{smallmatrix} h_1 & 0 \\ h_5 & h_4 \end{smallmatrix}\right),\ g_2 = \left(\begin{smallmatrix} h_1 & h_3 \\ 0 & h_2 \end{smallmatrix}\right),\ h_1 \in G_r,\ h_4 \in G_{k-r}, \right.$$
$$\left. h_2 \in G_{n-k-r},\ h_5 \in \mathrm{M}_{k-r\times r}(F),\ h_3 \in \mathrm{M}_{r\times n-k-r}(F) \right\}.$$

For $g = \left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right) \in M_r$ with $g_1 = \left(\begin{smallmatrix} h_1 & 0 \\ h_5 & h_4 \end{smallmatrix}\right)$ and $g_2 = \left(\begin{smallmatrix} h_1 & h_3 \\ 0 & h_2 \end{smallmatrix}\right)$, let

$$M_r \xrightarrow{p_1} P_{r,k}^*, \quad M_r \xrightarrow{p_2} P_{r,n-k}, \quad M_r \xrightarrow{p_3} G_r,$$

be the epimorphisms defined by

$$p_1(g) = g_1, \quad p_2(g) = g_2, \quad p_3(g) = h_1.$$

Let $d = \gcd(r,k,n)$ and let θ_r be a character of $F_r^\times$ with the property that $\theta_r|_{F_d^\times} = \theta|_{F_d^\times}$ (such a θ_r exists by part (1) of Lemma 2.1). Consider the representation $\mathrm{Ind}_{F_r^\times}^{G_r} \theta_r$ of G_r. The inflation of this representation to M_r via the epimorphism p_3 will be denoted by the same symbol $\mathrm{Ind}_{F_r^\times}^{G_r} \theta_r$. Let $\Pi_{r,n-k}$ and $\Pi_{r,k}^*$ be the representations of $P_{r,n-k}$ and $P_{r,k}^*$ respectively, defined in Section 4. The inflations of $\Pi_{r,n-k}$ and $\Pi^*_{r,k}$ to M_r via the epimorphisms p_2 and p_1 will be denoted by the same symbols.

Theorem 5.1. *Let $\pi = \pi_\theta$ be a cuspidal representation of G_n, and let P be a maximal parabolic subgroup of G associated with the partition $(k, n-k)$ of n. Let $P = MN$ be the Levi decomposition of P, and let ψ be a rank r character of N. Let θ_r be a character of $F_r^\times$ as above. The twisted Jacquet module $\pi_{N,\psi}$ as a representation of M_r is isomorphic to*

$$\pi_{N,\psi} \simeq \Pi^*_{r,k} \otimes \Pi_{r,n-k} \otimes \operatorname{Ind}_{F_r^\times}^{G_r} \theta_r.$$

In particular,

$$\dim \pi_{N,\psi} = q^{\binom{r}{2}} \prod_{i=1}^{r-1}(q^i - 1) \cdot \prod_{i=r}^{k-1}(q^i - 1) \cdot \prod_{i=r}^{n-k-1}(q^i - 1).$$

Proof. Let $\Theta_{N,\psi}$ denote the character of twisted Jacquet module $\pi_{N,\psi}$ of π_θ for a rank r character ψ of N. We recall that the character of π_θ equals the class function Θ_θ of Definition 2.2. By part (1) of Theorem 4.13, for $g' = \left(\begin{smallmatrix} g_1 & g_1 g_3 \\ & g_2 \end{smallmatrix}\right) \in P_{k,n}$, we have

$$\Theta_\theta(g') = \Theta_{k,n}(g') \cdot \Theta_{\theta_k}(g_1),$$

where we recall that θ_k is any character of $F_k^\times$ such that the restrictions to $F^\times_{\gcd\{k,n\}}$ of θ_k and θ are equal, and where Θ_{θ_k} is the class function (see Definition 2.2) associated to θ_k. Using this in Proposition 2.3 (with a rank r character ψ_r of N) for calculating the character $\Theta_{N,\psi}$ at $g = \left(\begin{smallmatrix} g_1 & \\ & g_2 \end{smallmatrix}\right) \in M_r$ with $g_1 = \left(\begin{smallmatrix} h_1 & 0 \\ h_5 & h_4 \end{smallmatrix}\right)$ and $g_2 = \left(\begin{smallmatrix} h_1 & h_3 \\ 0 & h_2 \end{smallmatrix}\right)$, we get:

$$\Theta_{N,\psi}(g) = \Theta_{\theta_k}(g_1) \cdot \chi_{(\Pi_{k,n})_r}(g).$$

By Definition 4.3, we have $(\Pi_{k,n})_r = \operatorname{St}_{G_r} \otimes \Pi_{r,n-k}$, and hence we get

$$\Theta_{N,\psi}(g) = [\Theta_{\theta_k}(g_1) \cdot \operatorname{St}_{G_r}(h_1)] \cdot \Theta_{r,n-k}(g_2).$$

Applying the result in equation (4.13) with $(k, n, \gcd\{k,n\}, \theta) \mapsto (r, k, \gcd\{r,k\}), \theta_k)$, we see that for $\tilde{\theta}$ any character of $F_r^\times$ such that the restrictions to $F^\times_{\gcd\{r,k\}}$ of $\tilde{\theta}$ and θ_k are equal, we have

$$\Theta_{\theta_k}(g_1) = \Theta^*_{r,k}(g_1) \cdot \Theta_{\tilde{\theta}}(h_1).$$

Using this in the previous equation, we get:

$$\Theta_{N,\psi}(g) = \left[\Theta_{\tilde{\theta}}(h_1) \cdot \operatorname{St}_{G_r}(h_1)\right] \cdot \Theta^*_{r,k}(g_1) \cdot \Theta_{r,n-k}(g_2).$$

By part (2) of Lemma 2.6, we can rewrite this as

$$\Theta_{N,\psi}(g) = \chi_{\operatorname{Ind}_{F_r^\times}^{G_r} \tilde{\theta}}(h_1) \cdot \Theta^*_{r,k}(g_1) \cdot \Theta_{r,n-k}(g_2).$$

The character of $\operatorname{Ind}_{F_r^\times}^{G_r} \beta$ at $h_1 \in G_r$ for any character β of $F_r^\times$, is zero unless h_1 comes from $F_r^\times$. Let $m | r$ be the degree of the minimal polynomial $p(X) \in F[X]$ of h_1. Since $g_2 = \left(\begin{smallmatrix} h_1 & h_3 \\ 0 & h_2 \end{smallmatrix}\right)$, we have by Theorem 4.9 (which applies as the semisimple part of h_1, which is h_1 itself, does come from $F_r^\times$), that $\Theta_{r,n-k}(g_2) = 0$ unless the semisimple part of g_2 comes from $F^\times_{n-k}$. By part (2) of Lemma 4.10, this implies that the degree m of $p(X)$ divides $\gcd\{r, n-k\}$. Repeating this argument with $g_1 = \left(\begin{smallmatrix} h_1 & 0 \\ h_5 & h_4 \end{smallmatrix}\right)$ and $\Theta^*_{r,k}(g_1)$, we see that $m | \gcd\{r,k\}$, and hence m divides $d = \gcd\{r,k,n\}$. In particular, the value of the character of $\operatorname{Ind}_{F_r^\times}^{G_r} \tilde{\theta}$ at h_1 only depends on $\tilde{\theta}|_{F_d^\times}$. Since $F_d \subset F_{\gcd\{r,k\}}$, and by definition of $\tilde{\theta}$, the restrictions to $F^\times_{\gcd\{r,k\}}$ of $\tilde{\theta}$ and θ are equal, we see that $\tilde{\theta}|_{F_d^\times} = \theta|_{F_d^\times} = \theta_r|_{F_d^\times}$, and hence, we may replace $\mathrm{Ind}_{F_r^\times}^{G_r} \tilde{\theta}$ with $\mathrm{Ind}_{F_r^\times}^{G_r} \theta_r$. Using this in the equation above, we conclude

$$\pi_{N,\psi} \simeq \Pi^*_{r,k} \otimes \Pi_{r,n-k} \otimes \mathrm{Ind}_{F_r^\times}^{G_r} \theta_r.$$

□

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