COMPUTING GAUSSIAN AND EXPONENTIAL INTEGRALS IN $\mathbb{R}^n$

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Abstract. We consider expectations of the type $\mathbf{E} \exp\left\{\sum_{i=1}^m \phi_i\right\}$, where $\phi_i : \mathbb{R}^n \longrightarrow \mathbb{C}$ are functions, each depending on a few coordinates of a point in $\mathbb{R}^n$, and the expectation is taken with respect to the standard Gaussian or symmetric exponential probability measures. We prove sufficient conditions, in terms of the Lipschitz constants of ϕ_i and the combinatorics of their dependencies, for the integral to be separated from 0, and, consequently, to be amenable to a computationally efficient approximation. We discuss applications to computing volumes of bodies and statistics on integer points in polyhedra in $\mathbb{R}^n$.

1. Introduction and main results

(1.1) The setup. Let $\mu = \mu_1 \times \cdots \times \mu_n$ be the product probability measure in $\mathbb{R}^n = \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ and let $\phi_1, \ldots, \phi_m : \mathbb{R}^n \longrightarrow \mathbb{C}$ be complex-valued random variables. We are interested in efficient computation (approximation) of the expectation

$$\mathbf{E} \exp\left\{\sum_{i=1}^m \phi_i\right\}. \tag{1.1.1}$$

Of course, as stated, the integral (1.1.1) is way too general. We will assume that each function ϕ_i depends only on a few coordinates of a point $x = (\xi_1, \ldots, \xi_n)$ and impose some restrictions on the dependencies between the functions. We will also control the Lipschitz constants of ϕ_i.

Integrals of the type (1.1.1) are ubiquitous in statistical physics and quantum field theory, see, for example, [Br86], [GJ87] and [FV18]. Here we are interested in the computational complexity issues, as well as some discrete geometry applications.

The problem of approximating (1.1.1) is very much related to the problem of deciding when (1.1.1) is not 0. We say that complex numbers $z_1 \neq 0$ and $z_2 \neq 0$

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approximate each other within a relative error of $\epsilon > 0$ if we can write $z_1 = e^{w_1}$ and $z_2 = e^{w_2}$ for some numbers $w_1, w_2 \in \mathbb{C}$ such that $|w_1 - w_2| \leq \epsilon$. It is immediately clear that having (1.1.1) equal 0 creates a difficulty: in any meaningful way, a relative approximation of 0 can only be 0.

As it turns out, having (1.1.1) separated from 0 opens a way to efficiently approximate the integral. This idea has been around for some time by now, see [Ba16] and [PR17]. Here we briefly sketch how it is done, providing more detail in Section 2.

We introduce a parameter $\lambda \in \mathbb{C}$ and, given functions $\phi_1, \ldots, \phi_m$, consider the expectation

$$F(\lambda) = \mathbf{E} \exp\left\{\lambda \sum_{i=1}^{n} \phi_i\right\} \tag{1.1.2}$$

as a function of λ. Let us fix a $\rho > 1$. Suppose that for some functions ϕ_i and some $M \geq 3$, we have

$$\frac{1}{M} \leq |F(\lambda)| \leq M \quad \text{provided} \quad |\lambda| < \rho. \tag{1.1.3}$$

It turns out then that one can approximate (1.1.1) within relative error $0 < \epsilon < 1$ from the moments

$$\mathbf{E} \left(\sum_{i=1}^{m} \phi_i\right)^k = \sum_{1 \leq i_1, \ldots, i_k \leq m} \mathbf{E} \left(\phi_{i_1} \cdots \phi_{i_k}\right) \quad \text{for} \quad k = O_\rho\left(\ln \ln M - \ln \epsilon\right), \tag{1.1.4}$$

where the implied constant in the "O" notation depends only on ρ. In some cases, (1.1.3) holds with $M = \exp\left\{(mn)^{O(1)}\right\}$,which results in the bound

$$k = O_\rho\left(\ln m + \ln n - \ln \epsilon\right)$$

in (1.1.4). We note that while the upper bound in (1.1.3) is usually quite straightforward, it is the lower bound there that requires work.

Assuming that each ϕ_i depends on $r_i = O(1)$ coordinates, computing each of the $m^{O(k)}$ expectations $\mathbf{E}\left(\phi_{i_1} \cdots \phi_{i_k}\right)$ in (1.1.4) reduces to integration in a coordinate subspace of dimension $r = r_1 + \ldots + r_k = O(k)$. In most applications, integration in a k-dimensional space can be done in $k^{O(k)}$ time, which in turn produces a quasi-polynomial algorithm of $(m+n)^{O_\rho(\ln((m+n)/\epsilon))}$ complexity to approximate (1.1.1).

In fact, to approximate (1.1.1), one can replace (1.1.3) by a weaker condition: it suffices to have the inequality satisfied for all λ in some fixed connected open set $\mathbb{U} \subset \mathbb{C}$ containing 0 and 1, and not necessarily in the disc $|z| < \rho$, cf. Section 2.2 of [Ba16].

Zeros of $F(\lambda)$ in (1.1.2) are of a considerable interest to statistical physics, as they correspond to phase transitions, with λ playing the role of the inverse temperature, see [FV18]. One can informally say that if the system stays sufficiently far away from a phase transition, then the partition function can be efficiently approximated. In probability, the absence of zeros is related to the Central Limit Theorem type behavior of sequences [MS26].

In this paper, we consider two special cases. In the first case, μ is the standard Gaussian measure with density

$$\frac{1}{(2\pi)^{n/2}} \exp\left\{ -\frac{1}{2} \sum_{i=1}^n \xi_i^2 \right\} \quad \text{where} \quad x = (\xi_1, \ldots, \xi_n), \tag{1.1.5}$$

and we control the Lipschitz constants of ϕ_i in the ℓ^2 norm. In the second case, μ is the symmetric exponential measure with density

$$\frac{1}{2^n} \exp\left\{ -\sum_{j=1}^n |\xi_j| \right\} \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n), \tag{1.1.6}$$

and we control the Lipschitz constants of ϕ_i in the ℓ^1 norm. In [Ba26], the author considered the integral (1.1.1) in the case of a general product probability measure μ in the product space $\Omega = \Omega_1 \times \cdots \times \Omega_n$, with controlled Lipschitz constants of ϕ_i in the Hamming metric of Ω.

To control the dependencies among ϕ_i, we introduce some formal definitions. We say that a function $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ *depends on the coordinates* $\{\xi_j : \ j \in J_\phi\}$ provided

$$\phi(\xi_1', \ldots, \xi_n') = \phi(\xi_1'', \ldots, \xi_n'') \quad \text{whenever} \quad \xi_j' = \xi_j'' \quad \text{for all} \quad j \in J_\phi,$$

and J_ϕ is the minimal set under inclusion with that property. We say that ϕ *depends on at most r coordinates* if $|J_\phi| \leq r$. We say that functions $\phi, \psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ *share a coordinate* if $J_\phi \cap J_\psi \neq \emptyset$.

A popular approach to analyze the integrals (1.1.1) and (1.1.2) is via the *cluster expansion*, see [Br86]. Here we pursue a different, inductive approach that seems to produce stronger results in terms of the combinatorics of dependencies and also in terms of the required analytic properties of functions ϕ_i. Thus, unlike in the case of the cluster expansion approach [Br86], we do not require to bound higher derivatives of ϕ_i or even to assume that the functions are smooth. The proofs for the Gaussian and exponential measures presented in this paper are quite similar and can be extended to other measures with sufficiently strong concentration properties and a suitable logarithmic Sobolev inequality, cf. Chapter 5 of [Le01].

Our first result deals with the Gaussian measure (1.1.5). We consider the standard ℓ^2 norm in $\mathbb{R}^n$:

$$\|x\|_2 = \left(\xi_1^2 + \ldots + \xi_n^2\right)^{1/2} \quad \text{where} \quad x = (\xi_1, \ldots, \xi_n).$$

For $L > 0$, a function $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ is *L-Lipschitz* in the ℓ^2 norm, provided

$$|\phi(x) - \phi(y)| \ \leq \ L\|x - y\|_2.$$

We prove the following result.

(1.2) Theorem. *Let $\phi_1, \ldots, \phi_m : \mathbb{R}^n \longrightarrow \mathbb{C}$ be 1-Lipschitz functions in the ℓ^2 norm. Suppose that*

(1) *For some $c \geq 2$ and all $j = 1, \ldots, n$, at most c functions ϕ_i depend on the coordinate ξ_j;*
(2) *For some $\Delta \geq 1$ and all $i = 1, \ldots, m$, the function ϕ_i shares a coordinate with at most Δ other functions ϕ_k and*
(3) *We have $(c-1)\Delta \geq 4$.*

Then for $\lambda \in \mathbb{C}$ such that

$$|\lambda| \ \leq \ \frac{1}{10\sqrt{(c-1)\Delta}}$$

we have

$$\left| \mathbf{E} \exp\left\{ \lambda \sum_{i=1}^m \phi_i \right\} \right| \ \geq \ \frac{1}{2^m} \mathbf{E} \left| \exp\left\{ \lambda \sum_{i=1}^m \phi_i \right\} \right|. \tag{1.2.1}$$

We are interested in the situations where the parameters c and Δ are small (fixed in advance), whereas m and n are allowed to grow.

Let functions $\phi_1, \ldots, \phi_m$ and parameters c and Δ be as in Theorem 1.2. Let us fix a $\rho > 1$. As we argue in Section 2, the expectation

$$\mathbf{E} \exp\left\{ \lambda \sum_{i=1}^m \phi_i \right\} \quad \text{where} \quad |\lambda| \ \leq \ \frac{1}{10\rho\sqrt{(c-1)\Delta}}$$

can be approximated within relative error ϵ in polynomial time from the moments (1.1.4) with $k = O_\rho\big(\ln((m+n)/\epsilon)\big)$, where the implied constant in the "O" notation depends only on ρ.

A weaker version of Theorem 1.2 is obtained in [Ba26]: there the functions ϕ_i are required to be 1-Lipschitz in the ℓ^1 norm, while the bound for $|\lambda|$ is inversely proportional to the product $c\sqrt{r}$, where each function ϕ_i depends on at most r coordinates and for any coordinate ξ_j there are at most c functions ϕ_i that depend on ξ_j.

We prove Theorem 1.2 in Section 5, and in Section 3 we describe some applications to integer point counting in polyhedra.

Our second result deals with the symmetric exponential measure (1.1.6). We consider the ℓ^1 norm in $\mathbb{R}^n$:

$$\|x\|_1 = |\xi_1| + \ldots + |\xi_n| \quad \text{where} \quad x = (\xi_1, \ldots, \xi_n).$$

For $L > 0$, a function $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ is *L-Lipschitz* in the ℓ^1 norm, provided

$$|\phi(x) - \phi(y)| \ \leq \ L\|x - y\|_1.$$

Our main result is as follows.

(1.3) Theorem. *Let $\phi_1, \ldots, \phi_m : \mathbb{R}^n \longrightarrow \mathbb{C}$ be 1-Lipschitz functions in the ℓ^1 norm. Suppose that*

(1) *For some $r \geq 9$, each function ϕ_i depends on at most r coordinates and*
(2) *For some $c \geq 13$ and each j, at most c functions ϕ_i depend on ξ_j.*

Then for all $\lambda \in \mathbb{C}$ such that

$$|\lambda| \leq \frac{1}{25(c-1)\sqrt{r}}$$

we have

$$\left| \mathbf{E} \exp\left\{ \lambda \sum_{i=1}^m \phi_i \right\} \right| \geq \frac{1}{2^m} \mathbf{E} \left| \exp\left\{ \lambda \sum_{i=1}^m \phi_i \right\} \right|. \tag{1.3.1}$$

Again, we are interested in the situations where r and c are small (fixed in advance), whereas m and n are allowed to grow.

Let functions ϕ_i and parameters r and c be as in Theorem 1.3. Let us fix a $\rho > 1$. Then

$$\mathbf{E} \exp\left\{ \lambda \sum_{i=1}^m \phi_i \right\} \quad \text{where} \quad |\lambda| \leq \frac{1}{25\rho(c-1)\sqrt{r}}$$

can be approximated within relative error ϵ in polynomial time from the moments (1.1.4) with $k = O_\rho\big(\ln((m+n)/\epsilon)\big)$,

We prove Theorem 1.3 in Section 6, and in Section 4 we describe some applications to computing volumes.

(1.4) Notation. We denote by $\mathbf{i}$ the imaginary unit, so $\mathbf{i}^2 = -1$. For a complex number $z = a + \mathbf{i}b$, we denote by $\Re$ the real part, and by $\Im$ the imaginary part: $\Re\, z = a$, $\Im\, z = b$.

2. Computing approximations

Here we discuss in some detail how Theorems 1.2 and 1.3 lead to approximation algorithms. For $\rho > 1$, let

$$\mathbb{D}_\rho = \{z : \ |z| < \rho\}$$

be an open disc of radius ρ in the complex plane.

(2.1) Approximating within an additive error. Let $f : \mathbb{D}_\rho \longrightarrow \mathbb{C}$ be a holomorphic function such that

$$|f(z)| \leq M \quad \text{for all} \quad z \in \mathbb{D}_\rho$$

and some $M > 1$. For an integer $k > 0$, let

$$T_k(f;z) = f(0) + \sum_{l=1}^{k} \frac{f^{(l)}(0)}{l!} z^l$$

be the Taylor polynomial of f of degree k computed at 0. The Cauchy bound

$$\left| \frac{f^l(0)}{l!} \right| \leq \frac{M}{\rho^l},$$

see, for example, Chapter 2 of [Ta19], implies that

$$(2.1.1) \qquad |f(1) - T_k(f;1)| = \left| \sum_{l=k+1}^{\infty} \frac{f^{(l)}(0)}{k!} \right| \leq M \sum_{l=k+1}^{\infty} \rho^{-l} = \frac{M}{(\rho-1)\rho^k}.$$

It follows from (2.1.1) that to approximate $f(1)$ by $T_k(f;1)$ within an additive error of $0 < \epsilon < 1$, it suffices to choose $k = O_\rho\big(\ln M - \ln \epsilon\big)$, where the implied constant in the "O" notation depends on ρ only.

(2.2) Approximating within a relative error. Suppose now that $g : \mathbb{D}_\rho \longrightarrow \mathbb{C}$ is a holomorphic function such that

$$(2.2.1) \qquad \frac{1}{M} \leq |g(z)| \leq M \quad \text{for all} \quad z \in \mathbb{D}_\rho$$

and some $M \geq 3$. Since $g(z) \neq 0$ for all $z \in \mathbb{D}_\rho$, we can choose a continuous branch of

$$f(z) = \ln g(z),$$

use the Taylor polynomial $T_k(f;z)$ of f to approximate $f(1)$ and then approximate $g(1)$ by $e^{f(1)}$. Since from (2.2.1) we have

$$|f(z)| \leq \ln M \quad \text{for all} \quad z \in \mathbb{D}_\rho,$$

it follows from (2.1.1) that to approximate $g(1)$ within relative error $0 < \epsilon < 1$, it suffices to choose

$$(2.2.2) \qquad k = O_\rho\left(\ln\ln M - \ln \epsilon\right).$$

Moreover, one can compute the derivatives $f^{(l)}(0)$ for $l = 1, \ldots, k$ from the derivatives $g^{(l)}(0)$ for $l = 1, \ldots, k$ in $O(k^2)$ time by solving a $k \times k$ non-degenerate triangular system of linear equations, so that the Taylor polynomial $T_k(f;z)$ of $f = \ln g$ can be computed in $O(k^2)$ time from the Taylor polynomial $T_k(g;z)$ of g. Indeed, since $f'(z) = g'(z)/g(z)$, we have $g'(z) = f'(z)g(z)$ and hence

$$g^{(k)}(0) = \sum_{l=0}^{k-1} \binom{k-1}{l} f^{(k-l)}(0) g^{(l)}(0),$$

see Section 2.2 of [Ba16] for detail.

We also remark that (2.2.1) can be replaced by a weaker condition that the inequalities hold for all z in some fixed connected open set $\mathbb{U} \subset \mathbb{C}$, containing 0 and 1. We can reduce this general case to that of the disc $\mathbb{D}_\rho$, by replacing $g(z)$ with the composition $g(\phi(z))$, where $\phi : \mathbb{D}_\rho \longrightarrow \mathbb{U}$ is a holomorphic map, such that $\phi(0) = 0$ and $\phi(1) = 1$, see Section 2.2 of [Ba16].

(2.3) Approximating expectations. Given functions $\phi_1, \ldots, \phi_m : \mathbb{R}^n \longrightarrow \mathbb{R}$, our goal is to approximate

$$F(\lambda) = \mathbf{E} \exp\left\{\lambda \sum_{i=1}^m \phi_i\right\}. \tag{2.3.1}$$

Without loss of generality, we assume that $\phi_i(0) = 0$, since adding a constant to ϕ_i, $\phi_i := \phi_i + a$, results in multiplying the integral (2.3.1) by $e^{\lambda a}$.

We assume that for some $\lambda_0 > 0$ the integral (2.3.1) converges absolutely and uniformly on compact subsets of the disc $|\lambda| < \lambda_0$ in the complex plane, and, moreover, that for some $M \geq 3$ we have

$$\frac{1}{M} \ \leq \ |F(\lambda)| \ \leq \ M \quad \text{provided} \quad |\lambda| < \lambda_0. \tag{2.3.2}$$

Let us fix some $\rho > 1$. To approximate $F(\lambda)$ for $\lambda \in \mathbb{C}$ such that $|\lambda| \leq \lambda_0/\rho$, we define $g_\lambda : \mathbb{D}_\rho \longrightarrow \mathbb{C}$ by

$$g_\lambda(z) = F(\lambda z).$$

Hence

$$\frac{1}{M} \ \leq \ |g_\lambda(z)| \ \leq \ M \quad \text{for all} \quad z \in \mathbb{D}_\rho.$$

It follows from Section 2.2, see (2.2.2) in particular, that one can approximate $F(\lambda) = g_\lambda(1)$ within relative error $0 < \epsilon < 1$ in polynomial time from the moments

$$g_\lambda^{(k)}(0) = \lambda^k \mathbf{E} \left(\sum_{i=1}^m \phi_i\right)^k \quad \text{for} \quad k = O_\rho\left(\ln\ln M - \ln \epsilon\right), \tag{2.3.3}$$

where the implied constant in the "O" notation depends on ρ only.

In the next two sections, we obtain bounds for M in the case of the standard Gaussian (Section 2.4) and symmetric exponential (Section 2.5) measures, in the context of Theorem 1.2 and Theorem 1.3 respectively.

(2.4) Gaussian measure. Let $\phi_1, \ldots, \phi_m : \mathbb{R}^n \longrightarrow \mathbb{C}$ be 1-Lipschitz functions in the ℓ^2 norm such that $\phi_i(0) = 0$ for $i = 1, \ldots, m$ and let $\lambda \in \mathbb{C}$ be as in Theorem 1.2. We define $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\psi(x) = \Re\left(\lambda \sum_{i=1}^m \phi_i(x)\right) \quad \text{for} \quad x \in \mathbb{R}^n,$$

so that

$$\mathbf{E}\left|\exp\left\{\lambda \sum_{i=1}^m \phi_i\right\}\right| = \mathbf{E}\, e^{\psi}, \tag{2.4.1}$$

where the expectation is taken with respect to the standard Gaussian measure with density (1.1.5). Since $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is m-Lipschitz and $\psi(0) = 0$, we have

$$-m\|x\|_2 \ \leq\ \psi(x)\ \leq\ m\|x\|_2 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Then,

$$\begin{aligned} \mathbf{E}\, e^{\psi} \ &\leq\ \mathbf{E}\, e^{m\|x\|_2} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left\{ m\|x\|_2 - \frac{1}{2}\|x\|_2^2 \right\}\, dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left\{ m\|x\|_2 - \frac{1}{4}\|x\|_2^2 \right\} e^{-\|x\|_2^2/4}\, dx \\ &\leq\ \frac{e^{m^2}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\|x\|^2/4}\, dx = e^{m^2} 2^{n/2} \end{aligned} \tag{2.4.2}$$

and

$$\begin{aligned} \mathbf{E}\, e^{\psi} \ &\geq\ \mathbf{E}\, e^{-m\|x\|_2} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left\{ -m\|x\|_2 - \frac{\|x\|_2^2}{2} \right\}\, dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left\{ -m\|x\|_2 + \frac{\|x\|_2^2}{4} \right\} e^{-3\|x\|_2^2/4}\, dx \\ &\geq\ \frac{e^{-m^2}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-3\|x\|_2^2/4}\, dx = e^{-m^2} \left(\frac{2}{3}\right)^{n/2}. \end{aligned} \tag{2.4.3}$$

It follows from Theorem 1.2 and (2.4.1)–(2.4.3) that in (2.3.2) we can choose $M = e^{m^2} 2^{n+m}$ and hence in (2.3.3) we have

$$k = O_\rho\left(\ln(m+n) - \ln \epsilon\right).$$

(2.5) Exponential measure. Let $\phi_1, \ldots, \phi_m : \mathbb{R}^n \longrightarrow \mathbb{C}$ be 1-Lipschitz functions in the ℓ^1 norm such that $\phi_i(0) = 0$ for $i = 1, \ldots, m$ and let $\lambda \in \mathbb{C}$ be as in Theorem 1.3. We define $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\psi(x) = \Re\left(\lambda \sum_{i=1}^m \phi_i(x)\right) \quad \text{for} \quad x \in \mathbb{R}^n,$$

so that

$$\mathbf{E}\left|\exp\left\{\lambda \sum_{i=1}^m \phi_i\right\}\right| = \mathbf{E}\, e^{\psi}, \tag{2.5.1}$$

where the expectation is taken with respect to the symmetric exponential measure with density (1.1.6). Clearly, ψ is $\frac{1}{2}$-Lipschitz in the ℓ^1 norm and since $\psi(0) = 0$, we have

$$-\frac{1}{2}\sum_{j=1}^n |\xi_j| \ \leq\ \psi(x)\ \leq\ \frac{1}{2}\sum_{j=1}^n |\xi_j| \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n).$$

Therefore,

$$\mathbf{E}\, e^{\psi} \ \leq \ \frac{1}{2^n} \int_{\mathbb{R}^n} \exp\left\{ -\frac{1}{2} \sum_{j=1}^n |\xi_j| \right\} \ dx = 2^n \tag{2.5.2}$$

and

$$\mathbf{E}\, e^{\psi} \ \geq \ \frac{1}{2^n} \int_{\mathbb{R}^n} \exp\left\{ -\frac{3}{2} \sum_{j=1}^n |\xi_j| \right\} \ dx = \left(\frac{2}{3}\right)^n. \tag{2.5.3}$$

It follows from (2.5.1)–(2.5.3) and Theorem 1.3 that in (2.3.2) we can choose $M = 3^n 2^m$ and hence in (2.3.3) we have

$$k = O_{\rho}\left(\ln(m+n) - \ln \epsilon\right).$$

3. Applications to integer points in polyhedra

(3.1) Integer points in polyhedra. Let $\mathbb{Z}_+^n \subset \mathbb{R}^n$ be the set of points with non-negative integer coordinates, let $A = (a_{ij})$ be a real $m \times n$ matrix and let $b = (b_1, \ldots, b_m)$ be a real m-vector. We define a polyhedron $P \subset \mathbb{R}^n$ by the system of linear equations and inequalities:

$$P = \Big\{ (\xi_1, \ldots, \xi_n) : \quad \sum_{j=1}^n a_{ij} \xi_j = b_i \quad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad \xi_j \ \geq \ 0 \quad \text{for} \quad j = 1, \ldots, n \Big\}. \tag{3.1.1}$$

Note that for any $s \neq 0$, the scaled matrix sA and vector sb define the same polyhedron P.

Suppose we want to compute the number $|P \cap \mathbb{Z}_+^n|$ of integer points in P. This is a well-known #P-hard problem, and even deciding whether $P \cap \mathbb{Z}_+^n \neq \emptyset$ is an NP-complete problem, so approximating $\left|P \cap \mathbb{Z}_+^n\right|$ is also computationally hard. Hence our goal is to find a reasonable computationally efficient relaxation of the problem.

Let $\mathbf{q} = (q_1, \ldots, q_n)$ be a vector of real numbers $0 < q_j < 1$ for $j = 1, \ldots, n$. We consider the multivariate geometric distribution in $\mathbb{Z}_+^n$ defined by

$$\mathbb{P}(x) = \prod_{j=1}^n (1 - q_j) q_j^{\xi_j} \quad \text{where} \quad x = (\xi_1, \ldots, \xi_n). \tag{3.1.2}$$

Note that

$$\mathbf{E}\, \xi_j = \frac{q_j}{1 - q_j} \quad \text{for} \quad j = 1, \ldots, n. \tag{3.1.3}$$

One reasonable choice of q_j is the *maximum entropy distribution*, constructed as follows. Suppose that the polyhedron P defined by (3.1.1) is bounded and has a non-empty relative interior, that is, contains a point $(\xi_1, \ldots, \xi_n)$ where $\xi_j > 0$ for all j. Then the strictly concave function

$$g(x) = \sum_{j=1}^{n} \big((\xi_j + 1)\ln(\xi_j + 1) - \xi_j \ln \xi_j\big) \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n)$$

attains it maximum on P at a unique point $z = (\zeta_1, \ldots, \zeta_n)$, which can be efficiently computed as a solution to a convex optimization problem. Let us now define

$$q_j = \frac{\zeta_j}{1 + \zeta_j} \quad \text{for} \quad j = 1, \ldots, n.$$

It follows from (3.1.3) that

$$\mathbf{E}\left(\sum_{j=1}^{n} a_{ij}\xi_j\right) = b_i \quad \text{for} \quad i = 1, \ldots, m,$$

so that the expectation of the random integer vector x defined by (3.1.2) lies in P. Moreover, it is proved in [BH10], see Theorem 4 there, that

$$\mathbb{P}(x) = e^{-g(z)} \quad \text{for all} \quad x \in P \cap \mathbb{Z}_+^n,$$

so that the probability mass function is constant on the set of integer points in P and hence

$$|P \cap \mathbb{Z}_+^n| = e^{g(z)}\mathbb{P}\left(P \cap \mathbb{Z}_+^n\right).$$

Thus counting integer points in P reduces to computing the probability

$$\mathbb{P}\left(P \cap \mathbb{Z}_+^n\right). \tag{3.1.4}$$

(3.2) Quadratic penalty function. We want to replace the probability (3.1.4) that is hard to compute by an easier computable statistics.

For a matrix A and a vector b in (3.1.1), we define $F_{A,b} : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ by

$$F_{A,b}(x) = \sum_{i=1}^{m}\left(-b_i + \sum_{j=1}^{n} a_{ij}\xi_j\right)^2 \quad \text{where} \quad x = (\xi_1, \ldots, \xi_n).$$

Hence $F_{A,b}(x) = 0$ if x satisfies the equations of (3.1.1) and $F_{A,b}(x) > 0$ if x does not. We then choose a vector $\mathbf{q} = (q_1, \ldots, q_n)$ where $0 < q_j < 1$ for $j = 1, \ldots, n$

and consider the expectation of $\exp\left\{-\frac{1}{2}F_{A,b}\right\}$ with respect to the multivariate geometric distribution (3.1.2):

$$\mathbf{E}\ \exp\left\{-\frac{1}{2}F_{A,b}\right\} = \Phi(A,b;\mathbf{q})\prod_{j=1}^{n}(1-q_j), \quad \text{where}$$

(3.2.1)
$$\Phi\big(A,b;\mathbf{q}\big) = \sum_{\substack{x\in\mathbb{Z}_+^n \\ x=(\xi_1,\ldots,\xi_n)}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\left(-b_i+\sum_{j=1}^{n}a_{ij}\xi_j\right)^2\right\}\prod_{j=1}^{n}q_j^{\xi_j}.$$

Clearly,

$$\Phi\big(A,b;\mathbf{q}\big)\prod_{j=1}^{n}(1-q_j) \ \geq\ \mathbb{P}\left(P\cap\mathbb{Z}^n\right),$$

so (3.2.1) provides an upper bound for (3.1.4), but unlike (3.1.4), it is amenable to an efficient approximation. We also note that

$$\lim_{s\longrightarrow+\infty}\Phi\big(sA,sb;\mathbf{q}\big)\prod_{j=1}^{n}(1-q_j) = \mathbb{P}\left(P\cap\mathbb{Z}^n\right),$$

that ought to impose some limits on the computability of $\Phi(sA,sb;\mathbf{q})$ for large scaling factors s.

Next, we fix A and b and consider $\Phi(A,b;\mathbf{q})$ defined by (3.2.1) as a function $\mathbf{q}\longmapsto\Phi(A,b;\mathbf{q})$ of $q_j\in\mathbb{C}$ satisfying

$$|q_j|<1 \quad \text{for} \quad j=1,\ldots,n,$$

where we agree that $q_j^0=1$. Clearly, $\Phi(A,b;\mathbf{q})$ is well-defined for such $\mathbf{q}$. It can be viewed as a partition function in the Potts model of a particular kind, cf. [FV18].

We prove the following result.

(3.3) Theorem. *Let $A=(a_{ij})$ be an $m\times n$ real matrix, let $b=(b_1,\ldots,b_m)$ be a real m-vector and let $\mathbf{q}=(q_1,\ldots,q_n)$ be a complex n-vector, such that $|q_j|<1$ for $j=1,\ldots,n$. Suppose that there are at most $r\geq 2$ non-zero entries in every row of A and at most $c\geq 1$ non-zero entries in every column of A. Suppose further that*

$$|b_i| \ \leq\ \frac{1}{10(r+1)\sqrt{c}} \quad \textit{for} \quad i=1,\ldots,m$$

and that

$$\frac{|q_j|}{1-|q_j|}\left(\sum_{i=1}^{m}a_{ij}^2\right)^{1/2} \ \leq\ \frac{1}{10(r+1)\sqrt{c}} \quad \textit{for} \quad j=1,\ldots,n.$$

Then for

$$\Phi(A, b; \mathbf{q}) = \sum_{\substack{x \in \mathbb{Z}_+^n \\ x=(\xi_1, \dots, \xi_n)}} \exp\left\{ -\frac{1}{2} \sum_{i=1}^m \left(-b_i + \sum_{j=1}^n a_{ij} \xi_j \right)^2 \right\} \prod_{j=1}^n q_j^{\xi_j},$$

we have

$$\frac{1}{2^{m+n}} \prod_{j=1}^n (1 - |q_j|) \ \leq \ |\Phi(A, b; q)| \ \leq \ \prod_{j=1}^n \frac{1}{1 - |q_j|}.$$

We note that given $\mathbf{q}$, we can always scale $A \longmapsto sA$ and $b \longmapsto sb$ for $s > 0$, so that the polyhedron P of (3.1.1) remains the same, while the scaled matrix sA and scaled vector sb satisfy the conditions of Theorem 3.3.

Before we prove Theorem 3.3, we discuss how it leads to an efficient algorithm to approximate $\Phi(A, b; \mathbf{q})$.

(3.4) Computing $\Phi(A, b, \mathbf{q})$**.** Let us fix $\rho > 1$ and suppose that a matrix $A = (a_{ij})$, a vector $b = (b_1, \dots, b_m)$ and a vector $\rho\mathbf{q} = (\rho q_1, \dots, \rho q_n)$ satisfy the conditions of Theorem 3.3. In particular, $|q_j| < \rho^{-1}$ for $j = 1, \dots, n$. Our goal is to approximate $\Phi(A, b; \mathbf{q})$.

Let $\mathbb{D}_\rho \subset \mathbb{C}$ be the open disc of radius ρ in the complex plane, centered at 0. For given A, b and $\mathbf{q}$, we define a function $g = g_{A,b,\mathbf{q}} : \mathbb{D}_\rho \longrightarrow \mathbb{C}$ by $g(z) = \Phi(A, b; z\mathbf{q})$. By Theorem 3.3, we have

$$\frac{1}{M} \ \leq |g(z)| \ \leq \ M \quad \text{for all} \quad z \in \mathbb{D}_\rho,$$

where we can choose

$$M = 2^{m+n} \prod_{j=1}^n \frac{1}{1 - |q_j|} \ < \ 2^{m+n} \left(\frac{\rho}{\rho - 1} \right)^n.$$

It follows then one can approximate the value of $g(1) = \Phi(A, b; \mathbf{q})$ within relative error $0 < \epsilon < 1$ in polynomial time from the derivatives $g^{(k)}(0)$ for

$$k = O_\rho \left(\ln(m + n) - \ln \epsilon \right), \tag{3.4.1}$$

cf. Section 2. We have

$$g^{(k)}(0) = k! \sum_{\substack{(\xi_1, \dots, \xi_n) \in \mathbb{Z}_+^n : \\ \xi_1 + \ldots + \xi_n = k}} \exp\left\{ -\frac{1}{2} \sum_{i=1}^m \left(-b_i + \sum_{j=1}^n a_{ij} \xi_j \right)^2 \right\} \prod_{j=1}^n q_j^{\xi_j},$$

and hence computing $g^{(k)}(0)$ reduces to the enumeration of $\binom{n+k-1}{k}$ non-negative integer solutions to the equation $\xi_1 + \ldots + \xi_n = k$.

In view of (3.4.1), we obtain a quasi-polynomial deterministic algorithm to approximate $\Phi(A, b; \mathbf{q})$.

To get a feeling of the statistics computed by $\Phi(A, b; \mathbf{q})$, consider a family of instances where the sparsity parameters r and c are fixed, while the dimensions m and n of the matrix $A = (a_{ij})$ and vector $b = (b_1, \ldots, b_m)$ are allowed to grow. We suppose further that the entries a_{ij} and b_i are integer and stay uniformly bounded, $a_{ij}, b_i = O(1)$, and that the probabilities $\mathbf{q} = (q_1, \ldots, q_n)$ remain separated from 1, which by (3.1.3) enforces $\mathbf{E}\, \xi_j = O(1)$, so that the expectations of the coordinates stay uniformly bounded. Then there is a scaling $A \longmapsto sA$, $b \longmapsto sb$ that makes A and b satisfy the conditions of Theorem 3.3 with some fixed slack $\rho > 1$ and hence makes $\Phi(A, b; \mathbf{q})$ efficiently computable, and such that the penalty $\left(b_i - \sum_{j=1}^n a_{ij}\xi_j\right)^2$ for violating an equation, when non-zero, is at least some positive constant $\Omega(1)$, depending on r and c alone. Hence the contribution of a random $x \in \mathbb{Z}_+^n$ to $\Phi(A, b; \mathbf{q})$ is exponentially small in the number of violated equations in (3.1.1).

The proof of Theorem 3.3 is based on Theorem 1.2 and a Fourier transform trick, also known in statistical physics as the *Hubbard - Stratonovich transformation*, which allows one to toggle between Gaussian and discrete partition functions, see Section 8.7.5 of [FV18].

(3.5) Lemma. *Given an $m \times n$ real matrix $A = (a_{ij})$, a real m-vector $b = (b_1, \ldots, b_m)$, and a complex n-vector $\mathbf{q} = (q_1, \ldots, q_n)$, where $|q_j| < 1$ for $j = 1, \ldots, n$, we define $f_j : \mathbb{R}^m \longrightarrow \mathbb{C}$, $j = 1, \ldots, n$, by*

$$f_j(t) = \left(1 - q_j \exp\left\{\mathbf{i} \sum_{i=1}^m a_{ij}\tau_i\right\}\right)^{-1} \quad \text{where} \quad t = (\tau_1, \ldots, \tau_m).$$

Then

$$\Phi(A, b; \mathbf{q}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left\{-\mathbf{i}\sum_{i=1}^m b_i\tau_i - \frac{1}{2}\sum_{i=1}^m \tau_i^2\right\} \prod_{j=1}^n f_j(t)\ dt.$$

Proof. We use the well-known formula

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\mathbf{i}\alpha\tau} e^{-\tau^2/2}\ d\tau = \exp\left\{-\frac{\alpha^2}{2}\right\}.$$

Let $t = (\tau_1, \ldots, \tau_m)$ be the standard Gaussian random m-vector and let $\mathbf{E}$ denote the expectation with respect to the standard Gaussian probability measure in $\mathbb{R}^m$. Then

$$\begin{aligned}
\Phi\big(A, b; \mathbf{q}\big) = & \sum_{\substack{x \in \mathbb{Z}_+^n \\ x=(\xi_1, \ldots, \xi_n)}} \exp\left\{-\frac{1}{2}\sum_{i=1}^m \left(-b_i + \sum_{j=1}^n a_{ij}\xi_j\right)^2\right\} \prod_{j=1}^n q_j^{\xi_j}. \\
= & \sum_{\substack{x \in \mathbb{Z}_+^n \\ x=(\xi_1, \ldots, \xi_n)}} \left(\mathbf{E} \exp\left\{\mathbf{i}\sum_{i=1}^m \tau_i \left(-b_i + \sum_{j=1}^n a_{ij}\xi_j\right)\right\}\right) \prod_{j=1}^n q_j^{\xi_j} \\
= & \mathbf{E}\left(\exp\left\{-\mathbf{i}\sum_{i=1}^m b_i\tau_i\right\}\right. \\
& \times \left. \sum_{\substack{x \in \mathbb{Z}_+^n \\ x=(\xi_1, \ldots, \xi_n)}} \exp\left\{\mathbf{i}\sum_{j=1}^n \xi_j \left(\sum_{i=1}^m a_{ij}\tau_i\right)\right\} \prod_{j=1}^n q_j^{\xi_j}\right) \\
= & \mathbf{E}\left(\exp\left\{-\mathbf{i}\sum_{j=1}^m b_i\tau_i\right\} \prod_{j=1}^n f_j(t)\right),
\end{aligned}$$

as required. □

(3.6) Proof of Theorem 3.3. We define functions $\phi_k : \mathbb{R}^m \longrightarrow \mathbb{C}$, $k = 1, \ldots, m+n$, as follows. For $t \in \mathbb{R}^m$, $t = (\tau_1, \ldots, \tau_m)$, we define

$$\phi_i(t) = \mathbf{i} b_i \tau_i \quad \text{for} \quad i = 1, \ldots, m$$

and

$$\phi_{m+j}(t) = \ln\left(1 - q_j \exp\left\{\mathbf{i}\sum_{i=1}^m a_{ij}\tau_i\right\}\right) \quad \text{for} \quad j = 1, \ldots, n.$$

Since $|q_j| < 1$, the functions ϕ_{m+j} are well-defined by the choice of the branch of the logarithm for $\phi_{m+j}(0) = \ln(1 - q_j)$. By Lemma 3.5,

$$\Phi(A, b; \mathbf{q}) = \mathbf{E} \exp\left\{-\sum_{i=1}^{m+n} \phi_i\right\}, \tag{3.6.1}$$

where the expectation is taken with respect to the standard Gaussian probability measure in $\mathbb{R}^m$.

To apply Theorem 1.2, we bound the Lipschitz constants of ϕ_i in the ℓ^2 norm of $\mathbb{R}^m$. Clearly, for $i = 1, \ldots, m$, the function ϕ_i is $|b_i|$-Lipschitz. Computing the gradient of ϕ_{m+j}, we get

$$\frac{\partial \phi_{m+j}}{\partial \tau_i} = -\mathbf{i} a_{ij} q_j \frac{\exp\left\{\mathbf{i}\sum_{i=1}^m a_{ij}\tau_i\right\}}{1 - q_j \exp\left\{\mathbf{i}\sum_{i=1}^m a_{ij}\tau_i\right\}} \quad \text{for} \quad j = 1, \ldots, n$$

and hence the Lipschitz constant of ϕ_{m+j} does not exceed

$$\|\nabla \phi_{m+j}\|_2 \ \leq \ \frac{|q_j|}{1-|q_j|} \left(\sum_{i=1}^m a_{ij}\right)^{1/2}.$$

Next, we observe that for each i, at most $r+1$ functions ϕ_k depend on the coordinate τ_i and that each function ϕ_k shares a coordinate with at most $(r+1)c$ other functions ϕ_i. Applying Theorem 1.2 to (3.6.1), we conclude that

$$|\Phi(A, b; \mathbf{q})| \ \geq \ \frac{1}{2^{m+n}} \mathbf{E} \left| \exp\left\{-\sum_{i=1}^{m+n} \phi_i\right\}\right|.$$

Now,

$$\mathbf{E} \left| \exp\left\{-\sum_{i=1}^{m+n} \phi_i\right\}\right| = \mathbf{E} \left| \exp\left\{-\sum_{j=1}^{n} \phi_{m+j}\right\}\right|. \tag{3.6.2}$$

We have

$$|\Re\, \phi_{m+j}(t)| \ \leq \ -\ln(1-|q_j|) \quad \text{for} \quad j=1, \ldots, n \tag{3.6.3}$$

and hence

$$|\Phi(A, b; \mathbf{q})| \ \geq \ \frac{1}{2^{m+n}} \prod_{j=1}^n (1-|q_j|).$$

Similarly, from (3.6.1) – (3.6.3), we obtain

$$|\Phi(A, b; \mathbf{q})| \ \leq \ \prod_{j=1}^n \frac{1}{1-|q_j|}.$$

□

4. Applications to volume computation

Randomized Markov Chain Monte Carlo algorithms have been spectacularly successful in efficiently approximating volumes of convex bodies, see [LV07]. Deterministic algorithms were noticeably less so, although there were some recent successful attempts for some special combinatorially defined polytopes [BR26], [GN25].

In this section, we apply Theorem 1.3 to approximate volumes of compact sets of a particular structure. Some of those sets are non-convex.

The application is based on a simple formula.

(4.1) Lemma. *Let $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ be a continuous function such that*

$$\Psi(\alpha x) = \alpha \Psi(x) \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and all} \quad \alpha \geq 0$$

and

$$\Psi(x) = 0 \Longrightarrow x = 0.$$

Then for the set $K_\Psi \subset \mathbb{R}^n$,

$$K_\Psi = \Big\{ x : \ \Psi(x) \leq 1 \Big\},$$

we have

$$\operatorname{vol} K_\Psi = \frac{1}{n!} \int_{\mathbb{R}^n} e^{-\Psi(x)} \ dx.$$

Proof. Clearly, K_Ψ is a compact set. For $t > 0$, we have

$$tK_\Psi = \Big\{ x : \ \Psi(x) \leq t \Big\}.$$

Let

$$F(t) = \operatorname{vol}\left(tK_\Psi\right) = t^n \operatorname{vol}(K) \quad \text{for} \quad t > 0.$$

Then

$$\int_{\mathbb{R}^n} e^{-\Psi(x)} \ dx = \int_0^{+\infty} e^{-t} \ dF(t) = \operatorname{vol}(K) \int_0^{+\infty} n t^{n-1} e^{-t} \ dt = n! \operatorname{vol}(K).$$

□

(4.2) Examples. Let us fix some $\rho > 1$. Suppose that

$$\Psi(x) = \sum_{j=1}^n |\xi_j| + \frac{\lambda}{\rho} \sum_{i=1}^m \phi_i(x) \quad \text{for} \quad x = \left(\xi_1, \ldots, \xi_n\right),$$

where $\lambda \in \mathbb{R}$ and $\phi_i : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ satisfy the conditions of Theorem 1.3 and, in addition, ϕ_i are positive homogeneous of degree 1:

$$\phi_i(\alpha x) = \alpha \phi_i(x) \quad \text{for} \quad \alpha \geq 0 \quad \text{and} \quad i = 1, \ldots, m.$$

By Lemma 4.1, we have

(4.2.1) $$\operatorname{vol} K_\Psi = \frac{1}{n!} \int_{\mathbb{R}^n} e^{-\Psi(x)} \ dx,$$

and as we discuss in Section 2, to approximate (4.2.1) within relative error $0 < \epsilon < 1$, one needs to compute $m^{O_\rho(\ln(m+n)-\ln \epsilon}$ integrals

$$\begin{aligned} & \int_{\mathbb{R}^n} \phi_{i_1}(x) \cdots \phi_{i_k}(x) \exp\left\{ -\sum_{j=1}^n |\xi_j| \right\} \, dx \\ & \text{for} \quad k = O_\rho\left(\ln(m+n) - \ln \epsilon\right). \end{aligned} \tag{4.2.2}$$

One natural example of such functions ϕ_i is provided by *support functions* of convex bodies, and, in particular, polytopes. Namely, let $B_i \subset \mathbb{R}^n$ be a convex body containing the origin in its relative interior, and suppose that

$$\phi_i(x) = \max_{y \in B_i} \langle x, y \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$. Clearly, ϕ_i is positive homogeneous of degree 1. Since the function

$$x \longrightarrow \sum_{j=1}^n |\xi_j| \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n)$$

is the support function of the cube $C_n = [-1, 1]^n$, then for $\lambda \geq 0$ the set K_Ψ is the polar of the Minkowski sum,

$$K_\Psi = \left(C_n + \frac{\lambda}{\rho} \sum_{i=1}^m B_i \right)^\circ.$$

If B_i lies in a coordinate subspace of dimension r_i then ϕ_i depends on at most r_i coordinates. Moreover, if B_i is a polytope defined as the convex hull of v_i vertices, then ϕ_i is a piece-wise linear functions, and the integrals (4.2.2) can be computed in $(rv)^{O(r)}$ time, where $r = r_1 + \ldots + r_m$ and $v_1 + \ldots + v_m$, see for example, [B+11].

Sets K_Ψ for which $\operatorname{vol} K_\Psi$ can be efficiently approximated via Theorem 1.3 include some non-convex sets. Suppose, for example, that $\Psi : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ is defined by

$$\Psi(x) = |\xi_1| + |\xi_2| - \epsilon|\xi_1 - \xi_2| \quad \text{where} \quad x = (\xi_1, \xi_2)$$

and $0 < \epsilon < 1$. Then the points $(1+\epsilon, 0)$ and $(0, 1+\epsilon)$ lie in K_Ψ but their average $\left(\frac{1+\epsilon}{2}, \frac{1+\epsilon}{2}\right)$ does not.

5. Proof of Theorem 1.2

To simplify notation, in this section we denote the ℓ^2 norm in $\mathbb{R}^n$ just by $\| \cdot \|$. We denote the standard Gaussian measure with density (1.1.5) by μ and denote the expectation with respect to that measure by $\mathbf{E}$. We say that a function $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ is L-Lipschitz, if it is L-Lipschitz in the ℓ^2 norm $\| \cdot \|$.

Our proof is based on the following well-known result regarding real-valued Lipschitz functions.

(5.1) Theorem. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be an L-Lipschitz function such that $\mathbf{E}\, f = 0$. Then*

(1) *We have*

$$\mathbf{E}\, e^f \ \le\ \exp\left\{\frac{L^2}{2}\right\};$$

(2) *For $a \ge 0$, we have*

$$\mu\Big\{x \in \mathbb{R}^n : \quad f(x) \ge a\Big\} \ \le\ \exp\left\{-\frac{a^2}{2L^2}\right\}.$$

Proof. See, for example, Section 5.1 of [Le01]. □

Next, we turn to complex-valued Lipschitz functions.

(5.2) Lemma. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be an L-Lipschitz function for $L = \frac{1}{4}$. Then*

$$\left|\mathbf{E}\, e^f\right| \ \ge\ \frac{1}{2}\mathbf{E}\, \left|e^f\right|.$$

Proof. Without loss of generality, we assume that $\mathbf{E}\, f = 0$. Let $f = g + \mathbf{i}h$, where $g, h : \mathbb{R}^n \longrightarrow \mathbb{R}$ are real-valued L-Lipschitz functions such that $\mathbf{E}\, g = \mathbf{E}\, h = 0$. From the Jensen inequality, we obtain

$$\mathbf{E}\, e^g \ \ge\ 1. \tag{5.2.1}$$

We have

$$e^{f(x)} = e^{g(x)}\left(\cos h(x) + \mathbf{i}\sin h(x)\right) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Let

$$X = \Big\{x \in \mathbb{R}^n : \quad |h(x)| \ \le\ 1\Big\} \quad \text{and} \quad \overline{X} = \mathbb{R}^n \setminus X.$$

From Part (2) of Lemma 5.1, we have

$$\mu(\overline{X}) \ \le\ 2\exp\left\{-\frac{1}{2L^2}\right\} = 2e^{-8}. \tag{5.2.2}$$

For all $x \in X$, we have $|h(x)| \le 1$ and hence

$$\begin{aligned} \left|\int_X e^f \ d\mu\right| \ &\ge \Re\left(\int_X e^f \ d\mu\right) = \int_X \Re\, e^f \ d\mu \ \ge\ \int_X (\cos 1) e^g \ d\mu \\ &= (\cos 1)\int_X e^g \ d\mu. \end{aligned} \tag{5.2.3}$$

Let $[\overline{X}]$ be the indicator of $\overline{X}$, that is, the function $[\overline{X}] : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$[\overline{X}](x) = \begin{cases} 1 & \text{if } x \notin X \\ 0 & \text{if } x \in X. \end{cases}$$

We use the Hölder inequality with

$$p = \frac{1}{L^2} = 16 \quad \text{and} \quad q = \frac{1}{1-L^2} = \frac{16}{15},$$

and get

$$\int_{\overline{X}} e^g \, d\mu = \mathbf{E} \left([\overline{X}] e^g \right) \ \leq \ \left(\mu(\overline{X}) \right)^{1/q} \left(\mathbf{E} \, e^{pg} \right)^{1/p}.$$

Since pg is pL-Lipschitz, from Part (1) of Lemma 5.1, we have

$$\mathbf{E} \, e^{pg} \ \leq \ \exp\left\{ \frac{p^2 L^2}{2} \right\}$$

and hence from (5.2.2) we obtain

(5.2.4)
$$\begin{aligned} \int_{\overline{X}} e^g \, d\mu \ &\leq \ \left(2 \exp\left\{ -\frac{1}{2L^2} \right\} \right)^{1/q} \exp\left\{ \frac{pL^2}{2} \right\} \\ &< \ 2 \exp\left\{ -\frac{1-L^2}{2L^2} + \frac{1}{2} \right\} = 2 \exp\left\{ 1 - \frac{1}{2L^2} \right\} = 2e^{-7}. \end{aligned}$$

Let

$$a = \int_X e^g \, d\mu \quad \text{and} \quad b = \int_{\overline{X}} e^g \, d\mu.$$

Clearly,

$$\mathbf{E} \, \left| e^f \right| = \mathbf{E} \, e^g = a + b$$

and from (5.2.3), we obtain

$$\left| \mathbf{E} \, e^f \right| \ \geq \ a(\cos 1) - b.$$

Therefore,

$$\frac{\left| \mathbf{E} \, e^f \right|}{\mathbf{E} \, \left| e^f \right|} \ \geq \ \frac{a(\cos 1) - b}{a + b} = \frac{(\cos 1) - (b/a)}{1 + (b/a)}.$$

From (5.2.1), we have $a + b \geq 1$ and hence $a \geq 1 - b$. Therefore,

$$\frac{b}{a} \ \leq \ \frac{b}{1-b}$$

and
$$\frac{\left|\mathbf{E}\, e^f\right|}{\mathbf{E}\, |e^f|} \ \geq \ \frac{(\cos 1) - b/(1-b)}{1 + b/(1-b)} = (\cos 1)(1-b) - b.$$

Since by (5.2.4), we have $b < 2e^{-7}$, we finally obtain

$$\frac{\left|\mathbf{E}\, e^f\right|}{\mathbf{E}\, |e^f|} \ \geq \ (\cos 1)\left(1 - 2e^{-7}\right) - 2e^{-7} \approx 0.5374931582 \ > \ 0.5,$$

as desired. □

Now we are ready to prove Theorem 1.2.

We fix c and Δ and will prove by induction on the number m of functions the following statement.

(5.3) Claim: The conclusion (1.2.1) of the theorem holds, and, moreover, the following holds.

Let $\phi_1, \ldots, \phi_m : \mathbb{R}^n \longrightarrow \mathbb{C}$ be functions as in Theorem 1.2, and let $\widehat{\phi}_m : \mathbb{R}^n \longrightarrow \mathbb{C}$ be yet another function, which is 1-Lipschitz and depends on a subset of the coordinates that ϕ_m depends on. Suppose further that

$$\left|\widehat{\phi}_m(x) - \phi_m(x)\right| \ \leq \ \tau \quad \text{for all} \quad x \in \mathbb{R}^n \tag{5.3.1}$$

and some $\tau > 0$. Then for $\lambda \in \mathbb{C}$ such that

$$|\lambda| \ \leq \ \frac{1}{10\sqrt{(c-1)\Delta}},$$

we have

$$\mathbf{E}\, \exp\left\{\lambda\left(\phi_m + \sum_{i=1}^{m-1} \phi_i\right)\right\} \neq 0, \quad \mathbf{E}\, \exp\left\{\lambda\left(\widehat{\phi}_m + \sum_{i=1}^{m-1} \phi_i\right)\right\} \neq 0$$

and the ratio of the two numbers can be written as e^α for some $\alpha \in \mathbb{C}$ such that

$$|\alpha| \ \leq \ 2|\lambda|\tau. \tag{5.3.2}$$

(5.4) Base $m = 1$. To simplify the notation somewhat, we drop the index 1 and denote the functions in question just by ϕ and $\widehat{\phi}$. Since functions $\lambda\phi$ and $\lambda\widehat{\phi}$ are L-Lipschitz for $L = 0.1 < 0.25$, by Lemma 5.2, we have

$$\left|\mathbf{E}\, e^{\lambda\phi}\right| \ \geq \ \frac{1}{2}\mathbf{E}\, \left|e^{\lambda\phi}\right| > 0 \quad \text{and} \quad \left|\mathbf{E}\, e^{\lambda\widehat{\phi}}\right| \ \geq \ \frac{1}{2}\mathbf{E}\, \left|e^{\lambda\widehat{\phi}}\right| > 0,$$

which establishes (1.2.1).

To establish (5.3.1)–(5.3.2), for $0 \leq s \leq 1$, let

$$\tilde{\phi}_s = (1-s)\phi + s\widehat{\phi}.$$

Hence $\tilde{\phi}_s$ is 1-Lipschitz, $\tilde{\phi}_0 = \phi$ and $\tilde{\phi}_1 = \widehat{\phi}$. In particular, $\mathbf{E}\, e^{\lambda \tilde{\phi}_s} \neq 0$ and we can choose a continuous branch of the function $s \longrightarrow \ln \mathbf{E}\, e^{\lambda \tilde{\phi}_s}$ in some neighborhood of the interval $[0,1] \subset \mathbb{C}$. Now,

$$\ln \mathbf{E}\, e^{\lambda \widehat{\phi}} - \ln \mathbf{E}\, e^{\lambda \phi} = \int_0^1 \left(\frac{d}{ds} \ln \mathbf{E}\, e^{\lambda \tilde{\phi}_s} \right) ds = \int_0^1 \lambda \frac{\mathbf{E}\, (\widehat{\phi} - \phi) e^{\lambda \tilde{\phi}_s}}{\mathbf{E}\, e^{\lambda \tilde{\phi}_s}}\, ds. \tag{5.4.1}$$

We have

$$\left| \mathbf{E}\, (\widehat{\phi} - \phi) e^{\lambda \tilde{\phi}_s} \right| \ \leq\ \mathbf{E}\, |\widehat{\phi} - \phi| \left| e^{\lambda \tilde{\phi}_s} \right| \ \leq\ \tau \mathbf{E} \left| e^{\lambda \tilde{\phi}_s} \right|.$$

Since $|\lambda| \leq 0.1 < 0.25$, by Lemma 5.2, we have

$$\left| \mathbf{E}\, e^{\lambda \tilde{\phi}_s} \right| \ \geq\ \frac{1}{2} \mathbf{E} \left| e^{\lambda \tilde{\phi}_s} \right|.$$

Therefore, from (5.4.1),

$$\left| \ln \mathbf{E}\, e^{\lambda \widehat{\phi}} - \ln \mathbf{E}\, e^{\lambda \phi} \right| \ \leq\ 2|\lambda|\tau,$$

and Claim 5.3 follows.

(5.5) Induction step $m-1 \Longrightarrow m$ for $m \geq 2$. Let $J \subset \{1, \ldots, n\}$ be the set of indices j such that ϕ_m depends on ξ_j and let $\overline{J} = \{1, \ldots, n\} \setminus J$ be its complement. We represent $\mathbb{R}^n$ as the direct sum $\mathbb{R}^n = \mathbb{R}^J \oplus \mathbb{R}^{\overline{J}}$. Let $I \subset \{1, \ldots, m-1\}$ be the set of indices i such that ϕ_i shares a coordinate with ϕ_m, that is, depends on some ξ_j with $j \in J$. Then $|I| \leq \Delta$. For a vector $x_J \in \mathbb{R}^J$ and a function ϕ_i with $i \in I$, we define a function $\phi_i(\cdot\ | x_J) : \mathbb{R}^{\overline{J}} \longrightarrow \mathbb{C}$ obtained by restricting the coordinates ξ_j with $j \in J$ to those of x_J. Further, we define $\Psi : \mathbb{R}^J \longrightarrow \mathbb{C}$ by

$$\Psi(x_J) = \mathbf{E}_{\overline{J}} \exp \left\{ \lambda \sum_{i \in I} \phi_i(\cdot\ | x_J) + \lambda \sum_{\substack{1 \leq i \leq m-1 \\ i \notin I}} \phi_i \right\}, \tag{5.5.1}$$

where the expectation is taken with respect to the coordinates ξ_j with $j \in \overline{J}$.

We compare values $\Psi(x'_J)$ and $\Psi(x''_J)$ for two vectors $x'_J, x''_J \in \mathbb{R}^J$. Switching from $x'_J = \left(\xi'_j\right)$ to $x''_J = \left(\xi''_j\right)$ in (5.5.1) affects at most Δ functions $\phi_i(\cdot\ | x_J)$ for $i \in I$. For $i \in I$, let $J_i \subset J$ be the set of indices $j \in J$ of the coordinates ξ_j shared by ϕ_i and ϕ_m, and let $x_{J,i}$ be the vector of the coordinates ξ_j for $j \in J_i$. We have

$$|\phi_i(\cdot\ | x'_J) - \phi_i(\cdot\ | x''_J)| \ \leq\ \left\| x'_{J,i} - x''_{J,i} \right\| \quad \text{for} \quad i \in I.$$

Applying the induction hypothesis, Claim 5.3, $|I|$ times, we conclude that $\Psi(x'_J) \neq 0$, $\Psi(x''_J) \neq 0$ and that we can write

$$\frac{\Psi(x'_J)}{\Psi(x''_J)} = e^{\alpha} \quad \text{for some} \quad \alpha \in \mathbb{C} \quad \text{such that}$$
$$|\alpha| \ \leq \ 2|\lambda| \sum_{i \in I} \left\| x'_{J,i} - x''_{J,i} \right\| \ \leq \ \frac{1}{5\sqrt{(c-1)\Delta}} \sum_{i \in I} \left\| x'_{J,i} - x''_{J,i} \right\|.$$

For $j \in J$, let $c_j \leq c-1$ be the number of functions ϕ_i for $i \in I$ that depend on ξ_j. By the Cauchy-Schwarz inequality,

$$\sum_{i \in I} \left\| x'_{J,i} - x''_{J,i} \right\| \ \leq \ \sqrt{|I|} \sqrt{\sum_{i \in I} \left\| x'_{J,i} - x''_{J,i} \right\|^2} \ \leq \ \sqrt{\Delta} \sqrt{\sum_{j \in J} c_j \left(\xi'_j - \xi''_j \right)^2}$$
$$\leq \ \sqrt{(c-1)\Delta} \left\| x'_J - x''_J \right\|.$$

Summarizing,

$$\frac{\Psi(x'_J)}{\Psi(x''_J)} = e^{\alpha} \quad \text{for some} \quad \alpha \in \mathbb{C} \quad \text{such that} \quad |\alpha| \ \leq \ \frac{1}{5} \left\| x'_J - x''_J \right\|.$$

It follows now that we can choose a continuous branch

$$\psi : \mathbb{R}^J \longrightarrow \mathbb{C}, \quad \psi(x_J) = \ln \Psi(x_J), \tag{5.5.2}$$

and that ψ is L-Lipschitz with $L = \frac{1}{5}$. We have

$$\begin{aligned} &\mathbf{E} \exp \left\{ \lambda \phi_m + \lambda \sum_{i=1}^{m-1} \phi_i \right\} = \mathbf{E}_J \exp \left\{ \lambda \phi_m + \psi \right\} \quad \text{and, similarly,} \\ &\mathbf{E} \exp \left\{ \lambda \widehat{\phi}_m + \lambda \sum_{i=1}^{m-1} \phi_i \right\} = \mathbf{E}_J \exp \left\{ \lambda \widehat{\phi}_m + \psi \right\}, \end{aligned} \tag{5.5.3}$$

where the expectations in the right hand side are taken with respect to the coordinates ξ_j with $j \in J$. The functions $\lambda \phi_m + \psi$ and $\lambda \widehat{\phi}_m + \psi$ are L-Lipschitz with

$$L = \frac{1}{5} + \frac{1}{10\sqrt{(c-1)\Delta}} \ \leq \ \frac{1}{4},$$

as we assumed that $(c-1)\Delta \geq 4$. Hence by Lemma 5.2 we have

$$\begin{aligned} &\left| \mathbf{E}_J \exp \left\{ \lambda \phi_m + \psi \right\} \right| \ \geq \ \frac{1}{2} \mathbf{E}_J \left| \exp \left\{ \lambda \phi_m + \psi \right\} \right| \quad \text{and, similarly,} \\ &\left| \mathbf{E}_J \exp \left\{ \lambda \widehat{\phi}_m + \psi \right\} \right| \ \geq \ \frac{1}{2} \mathbf{E}_J \left| \exp \left\{ \lambda \widehat{\phi}_m + \psi \right\} \right|. \end{aligned} \tag{5.5.4}$$

We now can establish the conclusion (1.2.1) of Theorem 1.2. Combining (5.5.3) and (5.5.4), we get

$$\left|\mathbf{E}\exp\left\{\lambda\sum_{i=1}^{m}\phi_i\right\}\right| = \left|\mathbf{E}_J e^{\lambda\phi_m+\psi}\right| \ \ge\ \frac{1}{2}\mathbf{E}_J\left|e^{\lambda\phi_m}\right|\left|e^{\psi}\right|. \tag{5.5.5}$$

On the other hand, using (5.5.2) and applying the induction hypothesis to Ψ, we conclude that

$$\begin{aligned}\left|e^{\psi(x_J)}\right| =& \left|\mathbf{E}_{\overline{J}}\exp\left\{\lambda\sum_{i\in I}\phi_i\left(\cdot\ |x_J\right)+\lambda\sum_{\substack{1\le i\le m-1\\ i\notin I}}\phi_i\right\}\right| \\ \ge& \frac{1}{2^{m-1}}\mathbf{E}_{\overline{J}}\left|\exp\left\{\lambda\sum_{i\in I}\phi_i\left(\cdot\ |x_J\right)+\lambda\sum_{\substack{1\le i\le m-1\\ i\notin I}}\phi_i\right\}\right|.\end{aligned} \tag{5.5.6}$$

Since ϕ_m does not depend on the coordinates ξ_j with $j\in\overline{J}$, from (5.5.5)–(5.5.6) we can further write

$$\begin{aligned}\left|\mathbf{E}\exp\left\{\lambda\sum_{i=1}^{m}\phi_i\right\}\right| \ \ge& \ \frac{1}{2^m}\mathbf{E}_J\mathbf{E}_{\overline{J}}\left|\exp\left\{\lambda\sum_{i\in I}\phi_i\left(\cdot\ |x_J\right)+\lambda\sum_{\substack{1\le i\le m\\ i\notin I}}\phi_i\right\}\right| \\ =&\frac{1}{2^m}\mathbf{E}\left|\exp\left\{\lambda\sum_{i=1}^{m}\phi_i\right\}\right|,\end{aligned}$$

which is the conclusion (1.2.1) of the theorem.

It remains to check (5.3.1)–(5.3.2). As in Section 5.4, for $0\le s\le 1$, we define

$$\tilde{\phi}_s=(1-s)\phi_m+s\widehat{\phi}_m,$$

so that $\tilde{\phi}_0=\phi_m$, $\tilde{\phi}_1=\widehat{\phi}_m$, and $\tilde{\phi}_s$ is 1-Lipschitz. As before, the function $\lambda\tilde{\phi}_s+\psi$ is L-Lipschitz with $L<0.25$ and hence $\mathbf{E}_J\exp\left\{\lambda\tilde{\phi}_s+\psi\right\}\ne 0$ and we can choose a continuous branch of the function $s\longmapsto\ln\mathbf{E}_J\exp\left\{\lambda\tilde{\phi}_s+\psi\right\}$ in some neighborhood of $[0,1]\subset\mathbb{C}$. Then

$$\begin{aligned}&\ln\mathbf{E}_J\exp\left\{\lambda\widehat{\phi}_m+\psi\right\}-\ln\mathbf{E}_J\exp\left\{\lambda\phi_m+\psi\right\}\\ &=\int_0^1\left(\frac{d}{ds}\ln\mathbf{E}_J\exp\left\{\lambda\tilde{\phi}_s+\psi\right\}\right)\,ds\\ &=\int_0^1\lambda\frac{\mathbf{E}_J(\widehat{\phi}_m-\phi_m)\exp\left\{\lambda\tilde{\phi}_s+\psi\right\}}{\mathbf{E}_J\exp\left\{\lambda\tilde{\phi}_s+\psi\right\}}\,ds.\end{aligned} \tag{5.5.7}$$

We have

$$\begin{aligned} \left|\mathbf{E}_J(\widehat{\phi}_m - \phi_m)\exp\left\{\lambda\tilde{\phi}_s + \psi\right\}\right| \ &\leq \ \mathbf{E}_J\left|\widehat{\phi}_m - \phi_m\right|\left|\exp\left\{\lambda\tilde{\phi}_s + \psi\right\}\right| \\ &\leq \ \tau\mathbf{E}_J\left|\exp\left\{\lambda\tilde{\phi}_s + \psi\right\}\right|. \end{aligned}$$

By Lemma 5.2,

$$\left|\mathbf{E}_J\exp\left\{\tilde{\phi}_s + \psi\right\}\right| \ \geq \ \frac{1}{2}\mathbf{E}_J\left|\exp\left\{\tilde{\phi}_s + \psi\right\}\right|,$$

and hence from (5.5.7)

$$\left|\ln\mathbf{E}_J\exp\left\{\lambda\widehat{\phi}_m + \psi\right\} - \ln\mathbf{E}_J\exp\left\{\lambda\phi_m + \psi\right\}\right| \ \leq \ 2|\lambda|\tau.$$

The proof of Claim 3.1 now follows by (5.5.3). □

6. Proof of Theorem 1.3

To simplify notation, in this section we denote the ℓ^1 norm in $\mathbb{R}^n$ just by $\|\cdot\|$. We denote the standard exponential measure with density (1.1.6) by μ and denote the expectation with respect to that measure by $\mathbf{E}$. We say that a function $f: \mathbb{R}^n \longrightarrow \mathbb{C}$ is L-Lipschitz, if it is L-Lipschitz in the ℓ^1 norm $\|\cdot\|$.

The proof is very similar to the proof of Theorem 1.2 in Section 5. First, we summarize some results regarding the symmetric exponential measure and real-valued Lipschitz functions.

(6.1) Theorem. *Suppose that $F: \mathbb{R}^n \longrightarrow \mathbb{R}$ is a differentiable function such that $\mathbf{E}\, F = 0$,*

$$\max_{j=1,\ldots,n}\left|\frac{\partial F}{\partial \xi_j}\right| \ \leq \ 1, \quad x = (\xi_1, \ldots, \xi_n),$$

and

$$\sum_{j=1}^{n}\left(\frac{\partial F}{\partial \xi_j}\right)^2 \ \leq \ a^2 \quad \text{for some} \quad a > 0.$$

Then

(1) *For $-\frac{1}{2} \ \leq \ \lambda \ \leq \ \frac{1}{2}$ we have*

$$\mathbf{E}\, e^{\lambda F} \ \leq \ \exp\left\{4a^2\lambda^2\right\};$$

(2) *For $r \geq 0$ we have*

$$\mu\left\{x \in \mathbb{R}^n: \quad F(x) \ \geq \ r\right\} \ \leq \ \exp\left\{-\frac{1}{4}\min\left\{r, \frac{r^2}{4a^2}\right\}\right\}.$$

Proof. See Section 5.3, in particular p. 105 of [Le01]. □

Next, we turn to complex-valued Lipschitz functions.

(6.2) Lemma. *Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ is L-Lipschitz for*

$$L = \min\left\{\frac{1}{12\sqrt{n}},\ \frac{1}{36}\right\}.$$

Then

$$\left|\mathbf{E}\, e^f\right| \ \geq\ \frac{1}{2}\mathbf{E}\,\left|e^f\right|.$$

Proof. A standard argument allows us to assume that f is differentiable. Indeed, for $\sigma > 0$, let $\psi_\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ be the Gaussian density with variance σ,

$$\psi_\sigma(x) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{j=1}^n \xi_j^2\right\} \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n),$$

and let $f_\sigma : \mathbb{R}^n \longrightarrow \mathbb{C}$ be the convolution of f and ψ_σ:

$$f_\sigma(x) = \int_{\mathbb{R}^n} f(x-y)\psi_\sigma(y)\ dy = \int_{\mathbb{R}^n} f(y)\psi_\sigma(x-y)\ dy.$$

From the first integral representation it follows that f_σ is L-Lipschitz and from the second representation it follows that f_σ is differentiable. In addition, $f_\sigma \longrightarrow f$ uniformly on compact sets, as $\sigma \longrightarrow 0+$.

Next, without loss of generality we assume that $\mathbf{E}\, f = 0$.

Let $f = g + \mathbf{i}h$, where $g, h : \mathbb{R}^n \longrightarrow \mathbb{R}$ are L-Lipschitz such that $\mathbf{E}\, g = \mathbf{E}\, h = 0$. From the Jensen inequality, we obtain

$$\mathbf{E}\, e^g \ \geq\ 1.$$

In addition,

$$\left|\frac{\partial g}{\partial \xi_j}\right|, \quad \left|\frac{\partial h}{\partial \xi_j}\right| \ \leq\ L \quad \text{for} \quad j = 1, \ldots, n.$$

Let

$$X = \bigl\{x \in \mathbb{R}^n : \quad |h(x)| \ \leq 1\ \bigr\} \quad \text{and} \quad \overline{X} = \mathbb{R}^n \setminus X.$$

Applying Part 2 of Theorem 6.1 with $F = L^{-1}h$, $a^2 = n$ and $r = L^{-1}$, we conclude that

$$\text{(6.2.1)} \qquad \mu\left(\overline{X}\right) \ \leq\ 2\exp\left\{-\frac{1}{4}\min\left\{\frac{1}{L},\ \frac{1}{4nL^2}\right\}\right\} \ \leq\ 2e^{-9}.$$

Denoting by $[\overline{X}]$ the indicator of $\overline{X}$ and using the Hölder inequality with

$$p = 18 \quad \text{and} \quad q = \frac{18}{17},$$

we get

$$(6.2.2)\quad \int_{\overline{X}} e^g \, d\mu = \mathbf{E}\left([\overline{X}]e^g\right) \ \le \ \left(\mathbf{E}\,[\overline{X}]\right)^{1/q} \left(\mathbf{E}\,e^{pg}\right)^{1/p} = \left(\mu\left(\overline{X}\right)\right)^{1/q} \left(\mathbf{E}\,e^{pg}\right)^{1/p}.$$

Applying Part 1 of Theorem 6.1 to $F = L^{-1}g$, $\lambda = pL = 18L$ and $a^2 = n$, we conclude that

$$(6.2.3)\qquad \left(\mathbf{E}\,e^{pg}\right)^{1/p} \ \le \ \exp\left\{4n \cdot 18 \cdot \frac{1}{12^2 n}\right\} = e^{1/2}.$$

Combining (6.2.1) - (6.2.3), we conclude that

$$(6.2.4)\qquad \int_{\overline{X}} e^g \, d\mu \ < 2\exp\left\{-\frac{17}{2} + \frac{1}{2}\right\} = 2e^{-8}.$$

On the other hand, for each $x \in X$, the argument of $e^{f(x)}$ does not exceed 1, and hence

$$\begin{aligned} \left|\int_X e^f \, d\mu\right| \ &\ge \Re \int_X e^f \, d\mu = \int_X \Re\, e^f \, d\mu \\ &\ge (\cos 1) \int_X e^g \, d\mu. \end{aligned}$$

The proof is finished as in Lemma 5.2. Let

$$a = \int_X e^g \, d\mu \quad \text{and} \quad b = \int_{\overline{X}} e^g \, d\mu.$$

As in the proof of Lemma 5.2, we have

$$\frac{\left|\mathbf{E}\,e^f\right|}{\mathbf{E}\,\left|e^f\right|} \ \ge \ (\cos 1)(1-b) - b.$$

Since by (6.2.4), we have $b < 2e^{-8}$, we conclude that

$$\frac{\left|\mathbf{E}\,e^f\right|}{\mathbf{E}\,\left|e^f\right|} \ \ge \ (\cos 1)\left(1 - 2e^{-8}\right) - 2e^{-8} \approx 0.539268878 \ > \ \frac{1}{2}.$$

□

Now we are ready to prove Theorem 1.3.

We fix r and c and prove by induction on the number m of functions the following statement.

(6.3) Claim: The conclusion (1.3.1) of Theorem 1.3 holds, and, moreover, the following holds.

Let $\phi_1, \ldots, \phi_m : \mathbb{R}^n \longrightarrow \mathbb{C}$ be functions as in Theorem 1.3, and let $\widehat{\phi}_m : \mathbb{R}^n \longrightarrow \mathbb{C}$ be yet another function, which is 1-Lipschitz and depends on a subset of the coordinates that ϕ_m depends on. Suppose further that

$$\left|\phi_m(x) - \widehat{\phi}_m(x)\right| \leq \tau \quad \text{for all} \quad x \in \mathbb{R}^n \tag{6.3.1}$$

and some $\tau > 0$. Then for all $\lambda \in \mathbb{C}$ such that

$$|\lambda| \leq \frac{1}{25(c-1)\sqrt{r}}.$$

we have

$$\mathbf{E} \exp\left\{\lambda\left(\widehat{\phi}_m + \sum_{i=1}^{m-1} \phi_i\right)\right\} \neq 0, \quad \mathbf{E} \exp\left\{\lambda\left(\phi_m + \sum_{i=1}^{m-1} \phi_i\right)\right\} \neq 0$$

and the ratio of the two numbers can be written as e^α for some $\alpha \in \mathbb{C}$ such that

$$|\alpha| \leq 2|\lambda|\tau. \tag{6.3.2}$$

(6.4) Base $m = 1$. We drop the index 1 and denote the functions in question just by ϕ and $\widehat{\phi}$ respectively. The coordinates that ϕ and $\widehat{\phi}$ depend on lie in a subset $\{\xi_j : \ j \in J\}$ with $|J| \leq r$, and hence without loss of generality we consider ϕ and $\widehat{\phi}$ as functions $\phi, \widehat{\phi} : \ \mathbb{R}^r \longrightarrow \mathbb{C}$.

Since function $\lambda\phi$ is L-Lipschitz with

$$L = \frac{1}{25(c-1)\sqrt{r}} < \frac{1}{12\sqrt{r}}$$

and since $r \geq 9$, by Lemma 6.2, we have

$$\left|\mathbf{E}\, e^{\lambda\phi}\right| \geq \frac{1}{2}\mathbf{E}\left|e^{\lambda\phi}\right|$$

and (1.3.1) follows.

Next, we prove (6.3.1)–(6.3.2). We define

$$\tilde{\phi}_s = (1-s)\phi + s\widehat{\phi} \quad \text{where} \quad 0 \leq s \leq 1,$$

so that $\tilde{\phi}_0 = \phi$ and $\tilde{\phi}_1 = \widehat{\phi}$. The functions $\tilde{\phi}_s$ are 1-Lipschitz. Since $|\lambda| < 1/12\sqrt{r}$ and $r \geq 9$, by Lemma 6.2, we have

$$\left|\mathbf{E}\, e^{\lambda\tilde{\phi}_s}\right| \geq \frac{1}{2}\mathbf{E}\left|e^{\lambda\tilde{\phi}_s}\right| \quad \text{for all} \quad 0 \leq s \leq 1. \tag{6.4.1}$$

Then $\mathbf{E}\, e^{\lambda \tilde{\phi}_s} \neq 0$ for all s in some neighborhood of the interval $[0,1] \subset \mathbb{C}$ and we can take a continuous branch of the function $s \longmapsto \ln \mathbf{E}\, e^{\lambda \tilde{\phi}_s}$ in that neighborhood. Then

$$\ln \mathbf{E}\, e^{\lambda \widehat{\phi}} - \ln \mathbf{E}\, e^{\lambda \phi} = \int_0^1 \left(\frac{d}{ds} \ln \mathbf{E}\, e^{\lambda \tilde{\phi}_s} \right) ds = \int_0^1 \lambda \frac{\mathbf{E} \left(\left(\widehat{\phi} - \phi \right) e^{\lambda \tilde{\phi}_s} \right)}{\mathbf{E}\, e^{\lambda \tilde{\phi}_s}}\, ds. \tag{6.4.2}$$

We have

$$\left| \mathbf{E} \left(\left(\widehat{\phi} - \phi \right) e^{\lambda \tilde{\phi}_s} \right) \right| \ \leq\ \mathbf{E} \left| \left(\widehat{\phi} - \phi \right) e^{\lambda \tilde{\phi}_s} \right| \ \leq\ \tau \mathbf{E} \left| e^{\lambda \tilde{\phi}_s} \right|.$$

It then follows from (6.4.1) and (6.4.2) that

$$\left| \ln \mathbf{E}\, e^{\lambda \widehat{\phi}} - \ln \mathbf{E}\, e^{\lambda \phi} \right| \ \leq\ 2|\lambda|\tau$$

and (6.3.2) follows.

(6.5) Induction step $m-1 \Longrightarrow m$ for $m \geq 2$. Let $J \subset \{1, \ldots, n\}$ be the set of indices j such that ϕ_m depends on ξ_j, so that $|J| \leq r$, and let $\overline{J} = \{1, \ldots, \} \setminus J$ be its complement. As in Section 5, we represent $\mathbb{R}^n$ as the direct sum $\mathbb{R}^n = \mathbb{R}^J \oplus \mathbb{R}^{\overline{J}}$. Let $I \subset \{1, \ldots, m-1\}$ be the set of indices i such that ϕ_i shares a coordinate with ϕ_m, that is, depends on some ξ_j with $j \in J$. For a vector $x_J \in \mathbb{R}^J$ and a function ϕ_i with $i \in I$, we define a function $\phi(\cdot | x_J) : \mathbb{R}^{\overline{J}} \longrightarrow \mathbb{C}$ obtained by restricting the coordinates ξ_j with $j \in J$ to those of x_J. Further, we define $\Psi : \mathbb{R}^J \longrightarrow \mathbb{C}$ by

$$\Psi\left(x_J\right) = \mathbf{E}_{\overline{J}} \exp \left\{ \lambda \sum_{i \in I} \phi_i(\cdot | x_J) + \lambda \sum_{\substack{1 \leq i \leq m-1 \\ i \notin I}} \phi_i \right\}, \tag{6.5.1}$$

where the expectation is taken with respect to the measure μ in $\mathbb{R}^{\overline{J}}$.

We compare values $\Psi(x'_J)$ and $\Psi(x''_J)$ for $x'_J, x''_J \in \mathbb{R}^J$. Switching from $x'_J = \left(\xi'_j\right)$ to $x''_J = \left(\xi''_j\right)$ affects the the functions $\phi_i(\cdot|x_J)$ for $i \in I$. For $i \in I$, let $J_i \subset J$ be the set of indices $j \in J$ of the coordinates shared by ϕ_i and ϕ_m. We have

$$|\phi_i(\cdot|x'_J) - \phi_i(\cdot|x''_J)| \ \leq\ \left\| x'_{J,i} - x''_{J,i} \right\| \quad \text{for} \quad i \in I.$$

Applying the induction hypothesis $|I|$ times, we conclude that $\Psi(x'_J) \neq 0$, $\Psi(x''_J) \neq 0$ and that we can write

$$\frac{\Psi(x'_J)}{\Psi(x''_J)} = e^{\alpha}$$

such that

$$|\alpha| \ \leq\ 2|\lambda| \sum_{i \in I} \|x'_{J,i} - x''_{J,i}\| \ \leq\ \frac{2}{25(c-1)\sqrt{r}} \sum_{i \in I} \|x'_{J,i} - x''_{J,i}\|.$$

Since there are at most $c-1$ functions ϕ_i with $i \in I$ that depend on any particular variable ξ_j with $j \in J$, we have

$$\sum_{i \in I} \|x'_{J,i} - x''_{J,i}\| \ \leq \ (c-1)\|x'_J - x''_J\|.$$

Summarizing,

$$\frac{\Psi(x'_J)}{\Psi(x''_J)} = e^{\alpha} \quad \text{for some} \quad \alpha \in \mathbb{C} \quad \text{such that} \quad |\alpha| \leq \frac{2}{25\sqrt{r}}\|x'_J - x''_J\|.$$

It follows now that we can choose a continuous branch

$$\psi : \mathbb{R}^J \longrightarrow \mathbb{C}, \quad \psi_J(x_J) = \ln \Psi(x_J),$$

and that ψ is L-Lipschitz with $L = \frac{2}{25\sqrt{r}}$. We have

$$\begin{aligned} &\mathbf{E} \exp\left\{\lambda\phi_m + \lambda\sum_{i=1}^{m-1}\phi_i\right\} = \mathbf{E}_J \exp\left\{\lambda\phi_m + \psi\right\} \quad \text{and} \\ &\mathbf{E} \exp\left\{\lambda\widehat{\phi}_m + \lambda\sum_{i=1}^{m-1}\phi_i\right\} = \mathbf{E}_J \exp\left\{\lambda\widehat{\phi}_m + \psi\right\}, \end{aligned} \tag{6.5.2}$$

where the expectations in the right hand side are taken with respect to the coordinates ξ_j with $j \in J$. We define

$$\tilde{\phi}_s = (1-s)\phi_m + s\widehat{\phi}_m \quad \text{where} \quad 0 \ \leq \ s \ \leq \ 1,$$

so that $\tilde{\phi}_0 = \phi_m$ and $\tilde{\phi}_1 = \widehat{\phi}_m$. The function $\tilde{\phi}_s$ is 1-Lipschitz, and hence the function $\lambda\tilde{\phi}_s + \psi$ is L-Lipschitz with

$$L = |\lambda| + \frac{2}{25\sqrt{r}} \ \leq \ \frac{1}{25(c-1)\sqrt{r}} + \frac{2}{25\sqrt{r}} \ \leq \ \frac{1}{25 \cdot 12\sqrt{r}} + \frac{2}{25\sqrt{r}} = \frac{1}{12\sqrt{r}}.$$

Consequently,

$$\mathbf{E}_J \left|\exp\left\{\lambda\tilde{\phi}_s + \psi\right\}\right| \ < \ +\infty$$

Since $r \geq 9$, from Lemma 6.2, we infer

$$\left|\mathbf{E}_J \exp\left\{\lambda\tilde{\phi}_s + \psi\right\}\right| \ \geq \ \frac{1}{2}\mathbf{E}_J \left|\exp\left\{\lambda\tilde{\phi}_s + \psi\right\}\right|. \tag{6.5.3}$$

In particular, $\mathbf{E}_J \exp\left\{\tilde{\phi}_s + \psi\right\} \neq 0$ for all $0 \leq s \leq 1$ and hence for all s is some neighborhood of $[0,1] \subset \mathbb{C}$. Then we can choose a continuous branch of the function $s \longmapsto \ln \mathbf{E}_J \exp\left\{\tilde{\phi}_s + \psi\right\}$ in that neighborhood and write

$$\begin{aligned}
&\ln \mathbf{E}_J \exp\left\{\lambda \widehat{\phi}_m + \psi\right\} - \ln \mathbf{E}_J \exp\left\{\lambda \phi_m + \psi\right\} \\
&= \int_0^1 \left(\frac{d}{ds} \ln \mathbf{E}_J \exp\left\{\lambda \tilde{\phi}_s + \psi\right\}\right) ds \\
&= \int_0^1 \lambda \frac{\mathbf{E}_J \left(\left(\widehat{\phi}_m - \phi_m\right) \exp\left\{\lambda \tilde{\phi}_s + \psi\right\}\right)}{\mathbf{E}_J \exp\left\{\lambda \tilde{\phi}_s + \psi\right\}} \, ds.
\end{aligned} \tag{6.5.4}$$

We have

$$\begin{aligned}
&\left|\mathbf{E}_J \left(\left(\widehat{\phi}_m - \phi_m\right) \exp\left\{\lambda \tilde{\phi}_s + \psi\right\}\right)\right| \\
&\leq \mathbf{E}_J \left(\left|\widehat{\phi}_m - \phi_m\right| \left|\exp\left\{\lambda \tilde{\phi}_s + \psi\right\}\right|\right) \\
&\leq \tau \mathbf{E}_J \left|\exp\left\{\lambda \tilde{\phi}_s + \psi\right\}\right|.
\end{aligned} \tag{6.5.5}$$

Combining (6.5.3) – (6.5.5), we conclude that

$$\left|\ln \mathbf{E}_J \exp\left\{\lambda \widehat{\phi}_m + \psi\right\} - \ln \mathbf{E}_J \exp\left\{\lambda \phi_m + \psi\right\}\right| \leq 2|\lambda|\tau,$$

which concludes the proof of (6.3.1) – (6.3.2).

It remains to prove (1.3.1). From (6.5.2) and (6.5.3), we have

$$\begin{aligned}
\left|\mathbf{E} \exp\left\{\lambda \sum_{i=1}^m \phi_i\right\}\right| &= \left|\mathbf{E}_J e^{\lambda \phi_m + \psi}\right| \geq \frac{1}{2} \mathbf{E}_J \left|e^{\lambda \phi_m + \psi}\right| \\
&= \frac{1}{2} \mathbf{E}_J \left(|\Psi| \left|e^{\lambda \phi_m}\right|\right),
\end{aligned} \tag{6.5.6}$$

where Ψ is defined by (6.5.1). By the induction hypothesis, for any $x_J \in \mathbb{R}^J$, we have

$$\begin{aligned}
|\Psi(x_J)| &= \left|\mathbf{E}_{\overline{J}} \exp\left\{\lambda \sum_{i \in I} \phi_i(\cdot | x_J) + \lambda \sum_{\substack{1 \leq i \leq m-1 \\ i \notin I}} \phi_i\right\}\right| \\
&\geq \frac{1}{2^{m-1}} \mathbf{E}_{\overline{J}} \left|\exp\left\{\lambda \sum_{i \in I} \phi_i(\cdot | x_J) + \lambda \sum_{\substack{1 \leq i \leq m-1 \\ i \notin I}} \phi_i\right\}\right|.
\end{aligned} \tag{6.5.7}$$

Combining (6.5.6) and (6.5.7) and using that ϕ_m does not depend on ξ_j with $j \in \overline{J}$, we conclude that

$$\left| \mathbf{E} \exp\left\{ \lambda \sum_{i=1}^m \phi_i \right\} \right| \geq \frac{1}{2^m} \mathbf{E}_J \mathbf{E}_{\overline{J}} \left| \exp\left\{ \lambda\phi_m + \lambda \sum_{i \in I} \phi_i(\cdot | x_J) + \lambda \sum_{\substack{1 \leq i \leq m-1 \\ i \notin I}} \phi_i \right\} \right|$$

$$= \frac{1}{2^m} \mathbf{E} \left| \exp\left\{ \lambda \sum_{i=1}^m \phi_i \right\} \right|,$$

which proves (1.3.1). □

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