LONG-TIME ASYMPTOTICS OF THE AUTOCORRELATION FUNCTION OF THE TRANSVERSE ISING CHAIN AT THE CRITICAL MAGNETIC FIELD REVISITED

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Abstract. Following [DZ94], we analyze the long-time asymptotics of the autocorrelation function of the transverse Ising chain at the critical magnetic field (a special case of the spin-$\frac{1}{2}$ XY model in a magnetic field) via the associated Riemann–Hilbert problem. We refine the original Deift-Zhou's result by determining the subleading growing term in the asymptotics.

Keywords: Quantum spin chain; correlation function; Riemann-Hilbert problem; steepest-descent method

Contents

1 Introduction and Main Results

Following the work of Deift-Zhou [DZ94], we consider the 1D transverse Ising model at the critical transverse magnetic field. In this formulation, spins on a 1D lattice interact via nearest-neighbor exchange interaction along the x-axis and are subjected to a transverse magnetic field along the z-axis. The Hamiltonian for this model is given by

$$\mathcal{H} = -\frac{1}{2}\sum_{l\in\mathbb{Z}}\left(\sigma_l^x\sigma_{l+1}^x + \sigma_l^z\right),$$

where σ_l^x and σ_l^z are the spin Pauli matrices at the l-th site of the 1D lattice. The main object of study in [DZ94] is the autocorrelation function $\chi(t)$ between the first spin component

$\sigma_0^x(t) = e^{-\mathrm{i}\mathcal{H}t}\sigma_0^x(0)e^{\mathrm{i}\mathcal{H}t}$ after time t and the one at time 0, i.e.,

$$\chi(t) = \frac{\mathrm{Tr}\left(e^{-\beta\mathcal{H}}e^{-\mathrm{i}\mathcal{H}t}\sigma_0^x(0)e^{\mathrm{i}\mathcal{H}t}\sigma_0^x(0)\right)}{\mathrm{Tr}\left(e^{-\beta\mathcal{H}}\right)},$$

where β is the inverse temperature. More precisely, the long-time behavior of $\chi(t)$ is the main interest of Deift-Zhou. Prior to that, McCoy-Perk-Shrock [MPS83] showed that $\chi(t)$ is given by the Fredholm determinant

$$\chi(t) = e^{-t^2/2}\det\left(\mathcal{I} - \mathcal{K}_t\right),$$

where $\mathcal{K}_t$ is a trace-class operator acting on $L^2((-1,1))$ with kernel

$$K_t(z,w) = g(z)\frac{\sin(\mathrm{i}t(z-w))}{\pi(z-w)}, \quad z,w \in [-1,1].$$

Here, we set $g(z) = \tanh(\beta\sqrt{1-z^2}) > 0$ on $(-1,1)$, which fixes the branch of the square root. As explained in [IIKN90], using the general theory of integrable operators, see [IIKS90, KBI93, Dei99] for example, one can show that this operator is associated with the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 1. *Find a* 2×2 *matrix function* $\Psi(z,t)$ *such that*

1. $\Psi(z,t)$ *is analytic for* $z \in \mathbb{C}\backslash[-1,1]$;
2. *one-sided traces* $\Psi_\pm(z,t)$ *exist a.e. on* $(-1,1)$, *are bounded there, and satisfy*
$$\Psi_+(z,t) = \Psi_-(z,t)G_\Psi(z,t), \quad z \in (-1,1),$$
where $(-1,1)$ *is oriented from* -1 *to* 1 *and*
$$G_\Psi(z,t) = \begin{pmatrix} 1+g(z) & -g(z)e^{2zt} \\ g(z)e^{-2zt} & 1-g(z) \end{pmatrix}, \quad z \in (-1,1);$$
3. *it holds that* $\Psi(z,t) = I + z^{-1}\Psi_1(t) + O(z^{-2})$ *as* $z \to \infty$.

Throughout the article, when we discuss the Riemann-Hilbert problems, we may omit explicit mention of the t-dependence of functions if no confusion is likely. Let

$$\sigma(t) := \frac{t^2}{2} + \ln\chi(t). \tag{1.1}$$

The connection of the solution to RHP 1 and $\chi(t)$ is given by

$$\sigma'(t) = -2[\Psi_1(t)]_{11}. \tag{1.2}$$

Thus, by solving RHP 1 asymptotically for large t, one can find the long-time asymptotics of $\sigma'(t)$ from which Deift and Zhou obtained the following result.

Theorem (Deift-Zhou). *As* $t \to \infty$, *one has*

$$\chi(t) = \exp\left[\frac{t}{\pi}\int_{-1}^{1}\ln|\tanh\beta\zeta|d\zeta + O(\ln t)\right]. \tag{1.3}$$

Our goal is to find a more explicit expression of the error term $O(\ln t)$ in (1.3). This is the problem posed by Percy Deift as one of the open problems at the conference[1] in Seoul to celebrate his 80th birthday.

This type of problem has attracted significant attention in related settings. Indeed, let us recall that the 1D transverse Ising model at the critical transverse magnetic field is a

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[1] Universality, Nonlinearity, and Integrability, May 2025, KIAS

particular case of the spin-$\frac{1}{2}$ XY model in a magnetic field, see for example [Hen84] and references therein. The Hamiltonian for this model is given by

$$\mathcal{H}_{\mathrm{XY}} = -\frac{1}{2}\sum_{l\in\mathbb{Z}}\Big((1+\gamma)\sigma_l^x\sigma_{l+1}^x + (1-\gamma)\sigma_l^y\sigma_{l+1}^y\Big) - h\sum_{l\in\mathbb{Z}}\sigma_l^z,$$

where γ is the anisotropic parameter and h is the transverse magnetic field. Note that γ interpolates between the Ising case ($\gamma = 1$) and the XXO Heisenberg chain ($\gamma = 0$). When we consider $\gamma = 1$ and $h = 1$ (the critical transverse magnetic field), then we indeed have

$$\mathcal{H} = \frac{1}{2}\,\mathcal{H}_{\mathrm{XY}}\big|_{\gamma=1,h=1}\,.$$

The other extremum, $\gamma = 0$, has also been extensively studied. Analyzing the associated Riemann-Hilbert problem, Its-Izergin-Korepin-Slavnov [IIKS93] found the asymptotics of the temperature correlation function in the spacelike direction and the timelike direction for a moderate magnetic field $0 \leqslant h < 1$ including the subleading growing term, see [IIKS93, Equations (4) and (5)][2]. For the strong magnetic field $h > 1$, the timelike direction asymptotics was further studied by Jie, see [Jie98, Section 1.4].

A similar analysis was carried out by Its-Izergin-Korepin-Varzugin [IIKV92] to find the asymptotics of the two-point correlation function of an impenetrable Bose gas, where the role of an external magnetic field is played by the so-called chemical potential. Based on the sign of the chemical potential, there are three cases: (i) the case of negative chemical potential, (ii) the case of positive chemical potential in the spacelike direction, and (iii) the case of positive chemical potential in the timelike direction, see [IIKV92, Section 8] for the asymptotics of the two-point correlation function for each case[3]. In [IIKV92], the authors have found all the growing terms and even the constant terms.

In the present paper, we study RHP 1 of Deift and Zhou in more detail to obtain the subleading growing term of $\chi(t)$ as $t \to \infty$ from the analysis of the relevant small norm Riemann-Hilbert problem. The constant term has to be separately cared for, since the Riemann-Hilbert approach gives the asymptotics of $\sigma'(t)$ only, recall (1.1) and (1.2), and this is our future project. Our main result is formulated as the following theorem.

Theorem 1. *As $t \to \infty$, it holds for some constant C that*

$$\chi(t) = \exp\left[\frac{t}{\pi}\int_{-1}^{1}\ln|\tanh\beta\zeta|d\zeta + \frac{\ln^2(\tanh\beta)}{2\pi^2}\ln t + C + O\big(t^{-1/4}\big)\right].$$

For the above-described reasons, significant parts of the analysis in the present paper are based on that of [DZ94]. Hence, we repeatedly cite [DZ94] for proofs of certain statements. However, to point out the differences, we present all the steps of the calculations needed to show the subleading growing terms in the remaining sections.

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[2] The critical value of the magnetic field is $h = 2$ in [IIKS93] due to a different normalization of the Hamiltonian.

[3] One can make an analogy between the Bose gas paper [IIKV92] and the XXO Heisenberg chain papers [Jie98, IIKS93]; case (i) of [IIKV92] corresponds with [Jie98] and case (ii)-(iii) of [IIKV92] corresponds with [IIKS93].

2 Deformations of RHP 1

In what follows, $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For our first step, we transfer RHP 1 to the unit circle $S^1 = \{|w| = 1\}$. To this end, let

$$\hat{\Psi}(w) := \Psi(z(w)), \quad z(w) := \frac{w - w^{-1}}{2\mathrm{i}},$$

which is analytic in $\mathbb{C}\backslash S^1$. Since it must hold that $g(z(w)) \geqslant 0$ on S^1, we get that

$$g(z(w)) = \begin{cases} h(w), & w \in S^1 \cap \{\mathrm{Re}(w) \geqslant 0\}, \\ -h(w), & w \in S^1 \cap \{\mathrm{Re}(w) \leqslant 0\}, \end{cases}$$

where

$$h(w) := \tanh\left(\beta \frac{w + w^{-1}}{2}\right). \tag{2.1}$$

As standard, we orient S^1 counter-clockwise. Notice that $z(w)$ maps $S^1 \cap \{\mathrm{Re}(w) \leqslant 0\}$ into $[-1, 1]$, but oriented from 1 to -1. Thus, the jump matrix of $\hat{\Psi}(w)$ satisfies

$$\begin{aligned} G_{\hat{\Psi}}(w) &:= \left\{ \begin{matrix} G_\Psi(z(w)), & w \in S^1 \cap \{\mathrm{Re}(w) \geqslant 0\} \\ G_\Psi(z(w))^{-1}, & w \in S^1 \cap \{\mathrm{Re}(w) \leqslant 0\} \end{matrix} \right\} \\ &= \begin{pmatrix} 1 + h(w) & -h(w)e^{-\mathrm{i}t(w - w^{-1})} \\ h(w)e^{\mathrm{i}t(w - w^{-1})} & 1 - h(w) \end{pmatrix}, \quad w \in S^1. \end{aligned} \tag{2.2}$$

Therefore, $\hat{\Psi}(w)$ solves the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 2. *Find a 2×2 matrix function $\hat{\Psi}(w, t)$ such that*

1. *$\hat{\Psi}(w, t)$ is analytic in $\overline{\mathbb{C}}\backslash S^1$;*
2. *one-sided traces $\hat{\Psi}_\pm(w, t)$ exist a.e. on S^1, are bounded there, and satisfy*
$$\hat{\Psi}_+(w, t) = \hat{\Psi}_-(w, t) G_{\hat{\Psi}}(w, t), \quad w \in S^1,$$
where $G_{\hat{\Psi}}(w, t)$ is given by (2.2);
3. *it holds that $\hat{\Psi}(w, t) = I + 2\mathrm{i}w^{-1}\Psi_1(t) + \mathcal{O}(w^{-2})$ as $w \to \infty$.*

Let us point out that RHP 1 and RHP 2 are simultaneously solvable and the solutions are related via $\hat{\Psi}(w) = \Psi(z(w))$. One direction is obvious by construction. In the other direction, given a solution of RHP 2, one needs to observe that it must satisfy

$$\hat{\Psi}(w) = \hat{\Psi}(-1/w)$$

since $G_{\hat{\Psi}}(w) = G_{\hat{\Psi}}(-1/w)$ as $h(w) = -h(-1/w)$, see [DZ94, Lemma 2.17] (the ratio of these two functions is analytic across S^1 and is equal to I at infinity; the claim then follows from Liouville's theorem). This, in turn, allows one to define $\Psi(z)$ out of $\hat{\Psi}(w)$. In a similar vein, observe that $G_{\hat{\Psi}}(w) = \sigma_1 G_{\hat{\Psi}}(-w)\sigma_1$, which then induces the relation

$$\hat{\Psi}(w) = \sigma_1 \hat{\Psi}(-w)\sigma_1, \quad w \in \overline{\mathbb{C}}\backslash S^1. \tag{2.3}$$

Next, we make the diagonal entries of jump matrix to be oscillatory by defining

$$\Phi(w) := \begin{cases} \hat{\Psi}(w)e^{\mathrm{i}tw^{-1}\sigma_3}, & |w| > 1, \\ \hat{\Psi}(w)e^{-\mathrm{i}tw\sigma_3}, & |w| < 1. \end{cases} \tag{2.4}$$

Then, the jump matrix for $\Phi(w)$ is given by

$$G_\Phi(w) = e^{-\mathrm{i}tw^{-1}\sigma_3} G_{\hat{\Psi}}(w) e^{-\mathrm{i}tw\sigma_3} = \begin{pmatrix} (1+h(w))e^{-t\varphi(w)} & -h(w) \\ h(w) & (1-h(w))e^{t\varphi(w)} \end{pmatrix},$$

where $\varphi(w)$ is purely imaginary on S^1 and is given by

$$\varphi(w) := \mathrm{i}(w + w^{-1}). \tag{2.5}$$

Furthermore, RHP 2(3) becomes

$$\begin{aligned} \Phi(w,t) &= \left(I + \frac{2\mathrm{i}}{w}\Psi_1(t) + O\left(w^{-2}\right)\right)\left(I + \frac{\mathrm{i}t}{w}\sigma_3 + O\left(w^{-2}\right)\right) \\ &= I + w^{-1}\left(2\mathrm{i}\Psi_1(t) + \mathrm{i}t\sigma_3\right) + O\left(w^{-2}\right), \quad w \to \infty. \end{aligned}$$

Once one has an oscillatory jump matrix, the next standard step in non-linear steepest descent analysis is to open the lenses, see (2.7). However, one technical issue arises from the fact that $h(w)$ vanishes on the imaginary axis including at $\pm\mathrm{i} \in S^1$, which complicates the desired factorization. This leads us to deform the jump contour S^1 away from $\pm\mathrm{i}$. To this end, we introduce some notation. Let

$$\mathbb{D}_r(a) := \{w \in \mathbb{C} \,|\, |w-a| < r\} \quad \text{and} \quad \mathbb{D} := \mathbb{D}_1(0).$$

Also, we observe that $G_\Phi^{-1}(w)$ is analytic near S^1. Then, for $\rho \in (0,1)$ sufficiently small so that the disks $\mathbb{D}_\rho(\pm\mathrm{i})$ do not contain other zeros of $h(w)$ besides $\pm\mathrm{i}$, we define $\tilde{\Phi}(w)$ by

$$\tilde{\Phi}(w) := \begin{cases} \Phi(w)G_\Phi^{-1}(w), & w \in \mathbb{D}_\rho(\pm\mathrm{i}) \cap \mathbb{D}, \\ \Phi(w), & \text{elsewhere.} \end{cases} \tag{2.6}$$

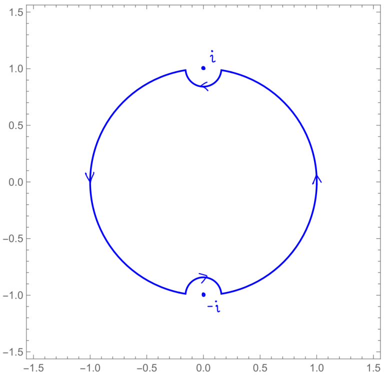


FIGURE 1. The jump contour Γ_{gl}.

Then, $\tilde{\Phi}(w)$ solves the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 3. *Find a* 2×2 *matrix function* $\tilde{\Phi}(w,t)$ *such that*

1. $\tilde{\Phi}(w,t)$ *is analytic in* $\mathbb{C}\setminus\Gamma_{gl}$*, where* Γ_{gl} *is an oriented contour as on Figure 1;*
2. *one-sided traces* $\tilde{\Phi}_\pm(w,t)$ *exist a.e. on* Γ_{gl}*, are bounded there, and satisfy*
$$\tilde{\Phi}_+(w,t) = \tilde{\Phi}_-(w,t)G_{\tilde{\Phi}}(w,t), \quad w \in \Gamma_{gl},$$
where
$$G_{\tilde{\Phi}}(w,t) = \begin{pmatrix} (1+h(w))e^{-t\varphi(w)} & -h(w) \\ h(w) & (1-h(w))e^{t\varphi(w)} \end{pmatrix};$$
3. *it holds that* $\tilde{\Phi}(w,t) = I + w^{-1}(2\mathrm{i}\Psi_1(t) + \mathrm{i}t\sigma_3) + O(w^{-2})$ *as* $w \to \infty$.

Clearly, RHP 2 and RHP 3 are simultaneously solvable as the transformation connecting them is invertible.

Next, we perform the Bruhat decomposition of $G_{\tilde\Phi}(w)$. Namely, it holds that

$$G_{\tilde\Phi}(w,t) = L_{ext}(w,t)P^+(w)L_{int}(w,t) = U_{ext}(w,t)P^-(w)U_{int}(w,t), \tag{2.7}$$

where

$$\begin{gathered} L_{ext}(w,t) = \begin{pmatrix} 1 & 0 \\ \frac{h(w)-1}{h(w)}e^{t\varphi(w)} & 1 \end{pmatrix}, \quad L_{int}(w,t) = \begin{pmatrix} 1 & 0 \\ \frac{h(w)+1}{-h(w)}e^{-t\varphi(w)} & 1 \end{pmatrix}, \\ P^+(w) = \begin{pmatrix} 0 & -h(w) \\ \frac{1}{h(w)} & 0 \end{pmatrix}, \quad P^-(w) = \begin{pmatrix} 0 & -\frac{1}{h(w)} \\ h(w) & 0 \end{pmatrix}, \\ U_{ext}(w,t) = \begin{pmatrix} 1 & \frac{h(w)+1}{h(w)}e^{-t\varphi(w)} \\ 0 & 1 \end{pmatrix}, \quad U_{int}(w,t) = \begin{pmatrix} 1 & \frac{h(w)-1}{-h(w)}e^{t\varphi(w)} \\ 0 & 1 \end{pmatrix}. \end{gathered} \tag{2.8}$$

It readily follows from the explicit expression (2.5) that

$$\begin{cases} \mathrm{Re}(\varphi(w)) > 0, & \{|w| < 1 \;\&\; \mathrm{Im}(w) > 0\} \cap \{|w| > 1 \;\&\; \mathrm{Im}(w) < 0\}, \\ \mathrm{Re}(\varphi(w)) < 0, & \{|w| < 1 \;\&\; \mathrm{Im}(w) < 0\} \cap \{|w| > 1 \;\&\; \mathrm{Im}(w) > 0\}. \end{cases} \tag{2.9}$$

Hence, we use the first factorization in (2.7) around $\Gamma_{gl}^+ := \Gamma_{gl} \cap \{\mathrm{Im}(w) \geqslant 0\}$ and we use the second factorization in (2.7) around $\Gamma_{gl}^- := \Gamma_{gl} \cap \{\mathrm{Im}(w) \leqslant 0\}$. To this end, let the curves $\Gamma_{int}^\pm, \Gamma_{ext}^\pm$ and the domains $\Omega_{int}^\pm, \Omega_{ext}^\pm$ be as on Figure 2. We assume that the curves $\Gamma_{int}^\pm, \Gamma_{ext}^\pm$ are close enough to Γ_{gl} so that the only zeros of $h(w)$ in $\Omega_{int}^\pm \cup \Omega_{ext}^\pm$ are $\pm$i. We also choose them so that $\Gamma_{ext}^- = \{-w | w \in \Gamma_{ext}^+\}$ and $\Gamma_{int}^- = \{-w | w \in \Gamma_{int}^+\}$. We set

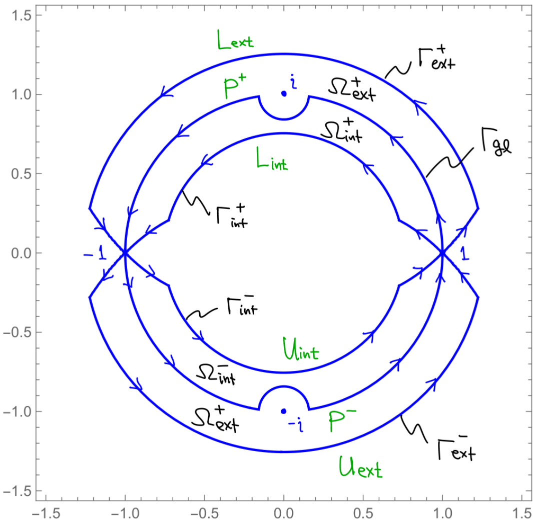


FIGURE 2. The jump contour $\Gamma := \Gamma_{gl} \cup \Gamma_{ext}^+ \cup \Gamma_{int}^+ \cup \Gamma_{ext}^- \cup \Gamma_{int}^-$ and the jump matrices $G_{\hat\Phi}(w,t)$ for RHP 4.

$$\hat\Phi(w) := \tilde\Phi(w) \begin{cases} L_{ext}(w,t), & w \in \Omega_{ext}^+, \\ L_{int}(w,t)^{-1}, & w \in \Omega_{int}^+, \\ U_{ext}(w,t), & w \in \Omega_{ext}^-, \\ U_{int}(w,t)^{-1}, & w \in \Omega_{int}^-, \\ I, & \text{otherwise}. \end{cases} \tag{2.10}$$

Since $1/h(w)$ has simple poles at $\pm\mathrm{i}$, each with residue $1/\beta$, $L_{ext}(w,t)$ and $U_{ext}(w,t)$ have simple poles at i and $-\mathrm{i}$, respectively, and so does $\hat{\Phi}(w)$. Indeed, we have that

$$L_{ext}(w) = \begin{pmatrix} 1 & 0 \\ \frac{h(w)-1}{h(w)} e^{t\varphi(w)} & 1 \end{pmatrix} = \frac{1}{w-\mathrm{i}} \begin{pmatrix} 0 & 0 \\ -\frac{1}{\beta} & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} + O((w-\mathrm{i}))$$

as $w \to \mathrm{i}$. Because $\hat{\Phi}(w) = \tilde{\Phi}(w)L_{ext}(w)$ around i, it holds that

$$\operatorname*{res}_{w=\mathrm{i}} \hat{\Phi}(w) = \tilde{\Phi}(\mathrm{i}) \begin{pmatrix} 0 & 0 \\ -\frac{1}{\beta} & 0 \end{pmatrix} = \lim_{w\to\mathrm{i}} \tilde{\Phi}(w) \begin{pmatrix} 0 & 0 \\ -\frac{1}{\beta} & 0 \end{pmatrix} = \lim_{w\to\mathrm{i}} \hat{\Phi}(w) \begin{pmatrix} 0 & 0 \\ -\frac{1}{\beta} & 0 \end{pmatrix}. \tag{2.11}$$

Similarly, at $w = -\mathrm{i}$, we have

$$\operatorname*{res}_{w=-\mathrm{i}} \hat{\Phi}(w) = \lim_{w\to-\mathrm{i}} \hat{\Phi}(w) \begin{pmatrix} 0 & \frac{1}{\beta} \\ 0 & 0 \end{pmatrix}. \tag{2.12}$$

Altogether, $\hat{\Phi}(w)$ is the solution of the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 4. *Find a* 2×2 *matrix function* $\hat{\Phi}(w,t)$ *such that*

(1) $\hat{\Phi}(w,t)$ *is analytic in* $\mathbb{C}\setminus(\Gamma \cup \{\pm\mathrm{i}\})$, *where* Γ *is an oriented contour from Figure 2;*
(2) *one-sided traces* $\hat{\Phi}_\pm(w,t)$ *exist a.e. on* Γ, *are bounded there, and satisfy*

$$\hat{\Phi}_+(w,t) = \hat{\Phi}_-(w,t)G_{\hat{\Phi}}(w,t), \qquad w \in \Gamma,$$

where the jump matrices $G_{\hat{\Phi}}(w,t)$ *on* Γ *are as in Figure 2;*
(3) *it holds that* $\hat{\Phi}(w,t) = I + w^{-1}(2\mathrm{i}\Psi_1(t) + \mathrm{i}t\sigma_3) + O(w^{-2})$ *as* $w \to \infty$ *by RHP 3(3);*
(4) $\hat{\Phi}(w,t)$ *has simple poles at* $w = \pm\mathrm{i}$ *with residues* (2.11) *and* (2.12).

Notice that the jump matrix $G_{\hat{\Phi}}(w,t)$ decays exponentially fast to the identity matrix on $\Gamma\setminus\Gamma_{gl}$ due to (2.9).

As before, RHP 3 and RHP 4 are simultaneously solvable. One direction is obvious. To go from RHP 4 to RHP 3 one needs to observe that

$$\hat{\Phi}(w)L_{ext}^{-1}(w) = \left(-\frac{1}{\beta}\begin{pmatrix} u & 0 \\ v & 0 \end{pmatrix}\frac{1}{w-\mathrm{i}} + \begin{pmatrix} * & u \\ * & v \end{pmatrix} + O((w-\mathrm{i}))\right) \times \left(\frac{1}{\beta}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\frac{1}{w-\mathrm{i}} + \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} + O((w-\mathrm{i}))\right) = O(1)$$

as $w \to \mathrm{i}$ by RHP 4(4) for some values u, v (these are the values of the second column of $\hat{\Phi}(w)$ at i). As an analogous computation holds around $-\mathrm{i}$, (2.10) is indeed reversible.

The jump matrix $G_{\hat{\Phi}}(w)$ on Γ_{gl} is independent of t, but is a function of w. In the next step, we turn it into a constant matrix. To this end, we set

$$Y(w) := \begin{cases} \hat{\Phi}(w)\delta(w)^{\sigma_3}, & w \in \mathrm{ext}(\Gamma_{gl}), \\ \hat{\Phi}(w)\delta(w)^{-\sigma_3}, & w \in \mathrm{int}(\Gamma_{gl}), \end{cases} \tag{2.13}$$

where $\mathrm{int}(\Gamma_{gl})$ and $\mathrm{ext}(\Gamma_{gl})$ are the interior and exterior domains of Γ_{gl}, respectively, and $\delta(w)$ is the solution of the following scalar Riemann-Hilbert problem.

Riemann-Hilbert Problem 5. *Find a scalar function* $\delta(w)$ *such that*

(1) $\delta(w)$ *is analytic in* $\mathbb{C}\setminus\Gamma_{gl}$*;*
(2) *one-sided traces* $\delta_\pm(w)$ *exist a.e. on* Γ_{gl}, *are bounded there, and satisfy*

$$\delta_+(w) = \delta_-(w)\begin{cases} h(w)^{-1}, & w \in \Gamma_{gl}^+, \\ h(w), & w \in \Gamma_{gl}^-; \end{cases}$$

(3) *it holds that* $\delta(w) \to 1$ *as* $w \to \infty$.

By the Plemelj-Sokhotski formula, the solution of RHP 5 is given by

$$\delta(w)=\exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{gl}^{+}}\frac{-\log h(s)}{s-w}ds+\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{gl}^{-}}\frac{\log h(s)}{s-w}ds\right\}, \tag{2.14}$$

where we take

$$\log h(s)=\begin{cases}\ln|h(s)|, & s\in S^1\cap\{\mathrm{Re}(s)>0\},\\ \ln|h(s)|-\mathrm{i}\pi, & s\in S^1\cap\{\mathrm{Re}(s)<0\ \&\ \mathrm{Im}(s)>0\},\\ \ln|h(s)|+\mathrm{i}\pi, & s\in S^1\cap\{\mathrm{Re}(s)<0\ \&\ \mathrm{Im}(s)<0\},\end{cases} \tag{2.15}$$

which ensures boundedness of $\delta(w)$ around ±1 (discussion following (4.4) further below implies that $|\delta(w)|$ is bounded above and away from zero in a neighborhood of 1, boundedness around -1 then follows from (2.16)). Since $|h(-s)|=|h(s)|$, one can use (2.14) and (2.15) to check that

$$\delta(w)\delta(-w)=\begin{cases}1, & w\in\mathrm{ext}(\Gamma_{gl}),\\ -1, & w\in\mathrm{int}(\Gamma_{gl}).\end{cases} \tag{2.16}$$

Moreover, it was shown in [DZ94, Equations (3.19) and (3.21)] that

$$\delta(\mathrm{i})=\sqrt{2\beta}\quad\text{and}\quad\delta(-\mathrm{i})=1/\sqrt{2\beta}. \tag{2.17}$$

Finally, $\delta(w)=1+w^{-1}\delta_1+O(w^{-2})$ as $w\to\infty$, where, see [DZ94, Equation (3.30)],

$$\delta_1=\frac{\mathrm{i}}{\pi}\int_{-1}^{1}\ln|\tanh\beta\zeta|d\zeta+\mathrm{i}. \tag{2.18}$$

Coming back to the matrix $Y(w)$, it readily follows from (2.13) and RHP 5(2) that $Y(w)$ admits the jump condition $Y_+(w)=Y_-(w)G_Y(w)$, where

$$G_Y(w)=\begin{cases}\delta(w)^{\sigma_3}G_{\hat\Phi}(w)\delta(w)^{-\sigma_3}, & w\in\Gamma_{int}^{+}\cup\Gamma_{int}^{-},\\ -\mathrm{i}\sigma_2, & w\in\Gamma_{gl},\\ \delta(w)^{-\sigma_3}G_{\hat\Phi}(w)\delta(w)^{\sigma_3}, & w\in\Gamma_{ext}^{+}\cup\Gamma_{ext}^{-}.\end{cases} \tag{2.19}$$

Furthermore, it follows from (2.13) and RHP 4(3) that

$$Y(w,t)=I+w^{-1}\big(2\mathrm{i}\Psi_1(t)+(\mathrm{i}t+\delta_1)\sigma_3\big)+O\big(w^{-2}\big)$$

as $w\to\infty$. Next, we get from (2.13), RHP 4(4), and (2.17) that

$$\begin{aligned}\operatorname*{res}_{w=\mathrm{i}}Y(w)&=\operatorname*{res}_{w=\mathrm{i}}\hat\Phi(w)\delta(\mathrm{i})^{\sigma_3}=\lim_{w\to\mathrm{i}}\hat\Phi(w)\begin{pmatrix}0&0\\-\frac{1}{\beta}&0\end{pmatrix}\delta(\mathrm{i})^{\sigma_3}\\&=\lim_{w\to\mathrm{i}}Y(w)\delta(\mathrm{i})^{-\sigma_3}\begin{pmatrix}0&0\\-\frac{1}{\beta}&0\end{pmatrix}\delta(\mathrm{i})^{\sigma_3}=\lim_{w\to\mathrm{i}}Y(w)\begin{pmatrix}0&0\\-2&0\end{pmatrix}.\end{aligned}$$

Clearly, the residue at $w=-\mathrm{i}$ can be computed similarly. Thus, $Y(w)$ is the solution of the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 6. *Find a 2×2 matrix function $Y(w,t)$ such that*

1. *$Y(w,t)$ is analytic in $\mathbb{C}\setminus(\Gamma\cup\{\pm\mathrm{i}\})$;*
2. *one-sided traces $Y_\pm(w)$ exist a.e. on Γ, are bounded there, and satisfy*

   $$Y_+(w,t)=Y_-(w,t)G_Y(w,t),\quad w\in\Gamma,$$

   where $G_Y(w,t)$ is given by (2.19);
3. *it holds that $Y(w,t)=I+w^{-1}Y_1(t)+O(w^{-2})$ as $w\to\infty$, where*

   $$Y_1(t)=2\mathrm{i}\Psi_1(t)+(\mathrm{i}t+\delta_1)\sigma_3;$$

(4) *$Y(w,t)$ has simple poles at $w=\pm\mathrm{i}$ with residues*

$$\operatorname*{res}_{w=\mathrm{i}} Y(w) = \lim_{w\to\mathrm{i}} Y(w)\begin{pmatrix}0 & 0\\ -2 & 0\end{pmatrix} \quad \text{and} \quad \operatorname*{res}_{w=-\mathrm{i}} Y(w) = \lim_{w\to-\mathrm{i}} Y(w)\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}.$$

Since the transformation (2.19) is clearly invertible, RHP 4 and RHP 6 are simultaneously solvable. For future use, observe that

$$G_Y(w) = \sigma_1 G_Y(-w)\sigma_1 \begin{cases} 1, & w\in\Gamma\backslash\Gamma_{gl},\\ -1, & w\in\Gamma_{gl},\end{cases} \tag{2.20}$$

by (2.8), (2.16), and (2.19). It is not immediate that (2.20) implies the corresponding symmetry of $Y(w)$ due to the presence of poles in RHP 6. However, we have already established that a solution RHP 6 produces a solution of RHP 2 by reversing (2.13), (2.10), (2.6), and (2.4). Then, (2.3) and (2.16) do give

$$Y(w) = \sigma_1 Y(-w)\sigma_1 \begin{cases} 1, & w\in\operatorname{ext}(\Gamma_{gl}),\\ -1, & w\in\operatorname{int}(\Gamma_{gl}).\end{cases} \tag{2.21}$$

It also follows from (1.2) and RHP 6(3) that

$$\sigma'(t) = \mathrm{i}\,[Y_1(t)]_{11} - \mathrm{i}\delta_1 + t. \tag{2.22}$$

3 Global Parametrix

As we mentioned before, the jump matrices in RHP 4 tend to the identity matrix as $t\to\infty$ on $\Gamma\backslash\Gamma_{gl}$. Moreover, we shall show later that $\delta(w)$ is bounded there. Hence, the leading order behavior of $Y(w)$ is determined by its jump on Γ_{gl}. Thus, we consider the following global parametrix Riemann-Hilbert problem.

Riemann-Hilbert Problem 7. *Find a 2×2 matrix function $P^{(gl)}(w)$ such that*

(1) *$P^{(gl)}(w)$ is analytic in $\overline{\mathbb{C}}\backslash\left(\Gamma_{gl}\cup\{\pm\mathrm{i}\}\right)$;*

(2) *one-sided traces $P^{(gl)}_\pm(w)$ exist a.e. on Γ_{gl}, are bounded there, and satisfy*

$$P^{(gl)}_+(w) = P^{(gl)}_-(w)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad w\in\Gamma_{gl};$$

(3) *it holds that $P^{(gl)}(w) = I + w^{-1}P^{(gl)}_1 + O(w^{-2})$ as $w\to\infty$;*

(4) *$P^{(gl)}(w)$ has simple poles at $w=\pm\mathrm{i}$ with residues*

$$\operatorname*{res}_{w=\mathrm{i}} P^{(gl)}(w) = \lim_{w\to\mathrm{i}} P^{(gl)}(w)\begin{pmatrix}0 & 0\\ -2 & 0\end{pmatrix},$$
$$\operatorname*{res}_{w=-\mathrm{i}} P^{(gl)}(w) = \lim_{w\to-\mathrm{i}} P^{(gl)}(w)\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}.$$

An explicit solution of RHP 7 is given by

$$P^{(gl)}(w) := A(w)B, \quad P^{(gl)}_1 = \begin{pmatrix}\mathrm{i} & 1\\ -1 & -\mathrm{i}\end{pmatrix}, \tag{3.1}$$

where

$$A(w) := \begin{pmatrix}\dfrac{w}{w-\mathrm{i}} & \dfrac{1}{w+\mathrm{i}}\\[2mm] \dfrac{-1}{w-\mathrm{i}} & \dfrac{w}{w+\mathrm{i}}\end{pmatrix} \quad \text{and} \quad B := \begin{cases} I, & w\in\operatorname{ext}(\Gamma_{gl}),\\ -\mathrm{i}\sigma_2, & w\in\operatorname{int}(\Gamma_{gl}).\end{cases} \tag{3.2}$$

One can readily check that

$$P^{(gl)}(w) = \sigma_1 P^{(gl)}(-w)\sigma_1 \begin{cases} 1, & w \in \operatorname{ext}(\Gamma_{gl}), \\ -1, & w \in \operatorname{int}(\Gamma_{gl}). \end{cases} \tag{3.3}$$

4 Approximate Local Parametrices

Since the jump matrices forming $G_Y(w,t)$ are not uniformly close to the identity matrix in a small disk around $w = \pm 1$, we need to solve RHP 6 locally in $\mathbb{D}_\epsilon(\pm 1)$ for some $\epsilon > 0$ fixed. There are many such solutions, as any of them can be multiplied by a holomorphic matrix function on the left. We seek ones that match well the global parametrix on $\partial\mathbb{D}_\epsilon(\pm 1)$.

4.1 Local Deformation of RHP 6

To construct approximate local solution of RHP 6 in $\mathbb{D}_\epsilon(1)$, we first remove the jump of $Y(w)$ on $\Gamma_{gl} \cap \mathbb{D}_\epsilon(1)$ by setting

$$\tilde{Y}(w,t) = Y(w,t)B^{-1}, \tag{4.1}$$

where B was defined in (3.2). For labeling reasons it will convenient for us to write

$$\begin{cases} \Gamma_0 = -\Gamma_{int}^- \cap \mathbb{D}_\epsilon(1), & \Gamma_2 = \Gamma_{ext}^+ \cap \mathbb{D}_\epsilon(1), \\ \Gamma_1 = -\Gamma_{ext}^- \cap \mathbb{D}_\epsilon(1), & \Gamma_3 = \Gamma_{int}^+ \cap \mathbb{D}_\epsilon(1) \end{cases}$$

(we flipped orientations of Γ_{int}^- and Γ_{ext}^-). Then, according to (2.8) and (2.19), $\tilde{Y}(w,t)$

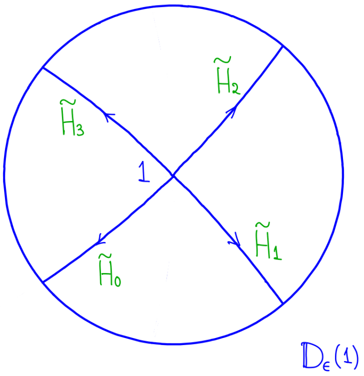


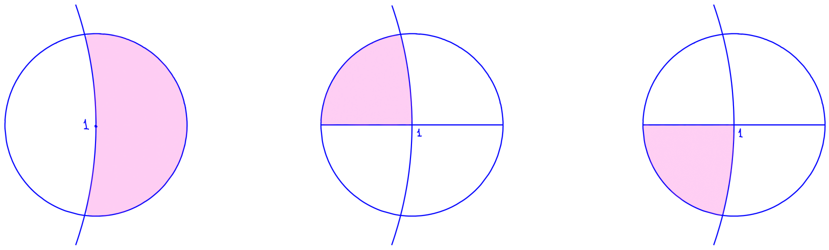
Figure 3. Arcs Γ_i and jump matrices $\tilde{H}_i(w,t)$ for $\tilde{Y}(w,t)$.

admits jump conditions as shown in Figure 3, where

$$\begin{aligned} \tilde{H}_0(w,t) &= \begin{pmatrix} 1 & 0 \\ \delta(w)^2\frac{1-h(w)}{h(w)}e^{t\varphi(w)} & 1 \end{pmatrix}, \quad \tilde{H}_1(w,t) = \begin{pmatrix} 1 & \frac{-1}{\delta(w)^2}\frac{h(w)+1}{h(w)}e^{-t\varphi(w)} \\ 0 & 1 \end{pmatrix}, \\ \tilde{H}_2(w,t) &= \begin{pmatrix} 1 & 0 \\ \delta(w)^2\frac{h(w)-1}{h(w)}e^{t\varphi(w)} & 1 \end{pmatrix}, \quad \tilde{H}_3(w,t) = \begin{pmatrix} 1 & \frac{1}{\delta(w)^2}\frac{h(w)+1}{h(w)}e^{-t\varphi(w)} \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{4.2}$$

To further transform $\tilde{Y}(w,t)$ near $w=1$, we study the behavior of $\delta(w)$ there. To this end, we define $K_{1,2,3}$ to be the subregion of $\mathbb{D}_\epsilon(1)$ as indicated on Figure 4, and set

$$\nu := \frac{1}{\pi \mathrm{i}} \ln h(1) \in \mathrm{i}\mathbb{R}, \quad \hat{h}(s) := \begin{cases} h(1)h(s)^{-1}, & s \in \Gamma_{gl}^+, \\ h(s)h(1)^{-1}, & s \in \Gamma_{gl}^-. \end{cases} \tag{4.3}$$



(A) K_1 (B) K_2 (C) K_3

FIGURE 4. Regions $K_{1,2,3}$.

Then, (2.14) can be written as

$$\begin{aligned} \delta(w) &= \exp\left\{\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{gl}} \frac{\log \hat{h}(s)}{s-w} ds\right\} \exp\left\{-\frac{\nu}{2} \int_{\Gamma_{gl}^+} \frac{ds}{s-w} + \frac{\nu}{2} \int_{\Gamma_{gl}^-} \frac{ds}{s-w}\right\} \\ &= \exp\left\{\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{gl}} \frac{\log \hat{h}(s)}{s-w} ds\right\} \left(\frac{w-1}{w+1}\right)_{\Gamma_{gl}^+}^{\nu/2} \left(\frac{w-1}{w+1}\right)_{\Gamma_{gl}^-}^{\nu/2}, \end{aligned}$$

where the power functions are positive for $w>1$ and subindex $\Gamma_{gl}^\pm$ indicates their respective branch-cuts. This means that

$$\left(\frac{w-1}{w+1}\right)_{\Gamma_{gl}^\pm}^{\nu/2} = \left(\frac{w-1}{w+1}\right)^{\nu/2} e^{\mp \pi \mathrm{i} \nu}, \quad w \in \{|w|<1\} \cap \{\pm \mathrm{Im}(w) > 0\},$$

where the root on the right-hand side of the above equality is principal. Thus,

$$\delta(w) = (w-1)^\nu \hat{\delta}(w) \begin{cases} 1, & w \in K_1, \\ e^{-\pi \mathrm{i} \nu}, & w \in K_2, \\ e^{\pi \mathrm{i} \nu}, & w \in K_3, \end{cases} \tag{4.4}$$

where the root is principal and $\hat{\delta}(w)$ is an analytic function $\mathbb{D}_\epsilon(1) \setminus \Gamma_{gl}$ given by

$$\hat{\delta}(w) = (w+1)^{-\nu} \exp\left\{\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{gl}} \frac{\log \hat{h}(s)}{s-w} ds\right\}.$$

Since ν is purely imaginary, $(w-1)^\nu$ is a bounded function around 1. We claim that $\hat{\delta}(w)$ is bounded there as well. We get from (4.3) that

$$(\log \hat{h}(s))' = \mp h'(s) h^{-1}(s), \quad s \in \Gamma_{gl}^\pm \setminus \{1\}.$$

Since $h'(1)=0$, $\log \hat{h}(s)$ is not only continuous on $\Gamma_{gl} \cap \mathbb{D}_\epsilon(1)$ (it is equal to 0 at 1), but is also continuously differentiable. Hence, $\hat{\delta}(w)$ extends continuously to $\Gamma_{gl} \cap \mathbb{D}_\epsilon(1)$. In fact, these extensions (from the left- and right-hand sides of Γ_{gl}) are continuously differentiable. Indeed, by [Gak90, Section 4.4] we need to show that the derivative extends smoothly to $\Gamma_{gl} \cap \mathbb{D}_\epsilon(1)$. It holds that

$$\hat{\delta}'(w) = \left(-\frac{\nu}{w+1} + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{gl}} \frac{\log \hat{h}(s)}{(s-w)^2} ds\right) \hat{\delta}(w).$$

Using integration by parts and (2.14) gives

$$\hat{\delta}'(w) = \left(-\frac{\nu}{w+1} + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{gl}} \frac{(\log \hat{h}(s))'}{s-w} ds \right) \hat{\delta}(w). \tag{4.5}$$

One can readily check that $\log \hat{h}(s)$ is, in fact, twice continuously differentiable on $(\Gamma_{gl} \backslash \{1\}) \cap \mathbb{D}_\epsilon(1)$ except for a jump discontinuity of the second derivative at 1. Hence, $(\log \hat{h}(s))'$ is Lipschitz continuous and respectively $\hat{\delta}'(w)$ extends smoothly to $\Gamma_{gl} \cap \mathbb{D}_\epsilon(1)$ (as a Hölder continuous function of any index < 1). Since $(\log \hat{h}(s))'$ vanishes at 1, this again implies that $\hat{\delta}'(1)$ is well defined.

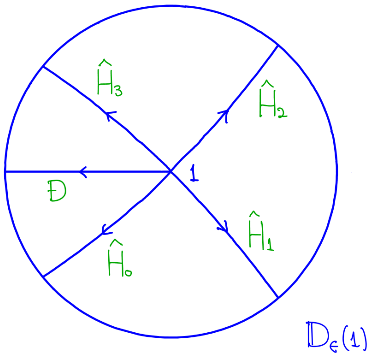


FIGURE 5. The jump contour and the jump matrices for $\hat{Y}(w,t)$.

Going back to the matrix $\tilde{Y}(w,t)$, let

$$\hat{Y}(w,t) = \tilde{Y}(w,t)(w-1)^{-\nu\sigma_3}, \quad w \in \mathbb{D}_\epsilon(1), \tag{4.6}$$

where the branch of the power function is principal. This transformation creates a jump across $(-1,1) \cap \mathbb{D}_\epsilon(1)$ while the jump matrices for $\tilde{Y}(w,t)$ are conjugated by $(w-1)^{\nu\sigma_3}$. More precisely, the jump matrices for $\hat{Y}(w,t)$ are shown on Figure 5, where

$$\begin{aligned}
\hat{H}_0(w,t) &= \begin{pmatrix} 1 & 0 \\ \hat{\delta}(w)^2 e^{2\pi \mathrm{i}\nu} \frac{1-h(w)}{h(w)} e^{t\varphi(w)} & 1 \end{pmatrix}, \quad w \in \Gamma_0, \\
\hat{H}_1(w,t) &= \begin{pmatrix} 1 & -\frac{1}{\hat{\delta}(w)^2} \frac{h(w)+1}{h(w)} e^{-t\varphi(w)} \\ 0 & 1 \end{pmatrix}, \quad w \in \Gamma_1, \\
\hat{H}_2(w,t) &= \begin{pmatrix} 1 & 0 \\ \hat{\delta}(w)^2 \frac{h(w)-1}{h(w)} e^{t\varphi(w)} & 1 \end{pmatrix}, \quad w \in \Gamma_2, \\
\hat{H}_3(w,t) &= \begin{pmatrix} 1 & \frac{1}{\hat{\delta}(w)^2} e^{2\pi \mathrm{i}\nu} \frac{h(w)+1}{h(w)} e^{-t\varphi(w)} \\ 0 & 1 \end{pmatrix}, \quad w \in \Gamma_3, \\
D &= e^{2\pi \mathrm{i}\nu\sigma_3}, \quad w \in (1-\epsilon, 1),
\end{aligned}$$

by (4.2), (4.4), and (4.6) (notice that we oriented the segment $(1-\epsilon, 1)$ away from 1).

4.2 Parabolic Cylinder Model Riemann-Hilbert Problem

It readily follows from (2.5) that

$$\varphi(w) = 2\mathrm{i} + \mathrm{i}\frac{(w-1)^2}{w},$$

where the second summand can be interpreted as a square of a conformal map, see (4.12). Then, by replacing $\hat{\delta}(w)$, $h(w)$ by $\hat{\delta}(1)$, $h(1)$, and $t\varphi(w)$ by $2it - \zeta^2/2$, we formally arrive at the following model Riemann-Hilbert problem.

Riemann-Hilbert Problem 8. *Find a* 2×2 *matrix function* $Z(\zeta)$ *such that*

(1) $Z(\zeta)$ *is analytic for* $\zeta \in \mathbb{C}\backslash\Gamma_Z$, *where* Γ_Z *is depicted in Figure 6;*
(2) $Z(\zeta)$ *has continuous traces on* $\Gamma_Z\backslash\{0\}$ *that satisfy*

$$Z_+(\zeta) = Z_-(\zeta)G_Z(\zeta), \quad \zeta \in \Gamma_Z,$$

where the jump matrices comprising $G_Z(\zeta)$ *are as on Figure 6 and given by*[4]

$$\begin{aligned} &H_0^Z(\zeta) = \begin{pmatrix} 1 & 0 \\ -s_2 e^{2\pi i\nu} e^{-\zeta^2/2} & 1 \end{pmatrix}, \quad H_1^Z(\zeta) = \begin{pmatrix} 1 & s_1 e^{\zeta^2/2} \\ 0 & 1 \end{pmatrix}, \\ &H_2^Z(\zeta) = \begin{pmatrix} 1 & 0 \\ s_2 e^{-\zeta^2/2} & 1 \end{pmatrix}, \quad H_3^Z(\zeta) = \begin{pmatrix} 1 & -s_1 e^{2\pi i\nu} e^{\zeta^2/2} \\ 0 & 1 \end{pmatrix}, \\ &D^Z = e^{2\pi i\nu\sigma_3}, \quad s_1 = -\frac{1}{\hat{\delta}(1)^2}\frac{h(1)+1}{h(1)}e^{-2it}, \quad s_2 = \hat{\delta}(1)^2\frac{h(1)-1}{h(1)}e^{2it}; \end{aligned} \tag{4.7}$$

(3) *it holds that* $Z(\zeta) = (I + O(1/\zeta))\zeta^{-\nu\sigma_3}$, $\zeta \to \infty$, *where* $\zeta^\nu > 0$ *when* $\zeta > 0$ *and* $\arg(\zeta) \in (-\pi/4, 7\pi/4)$.

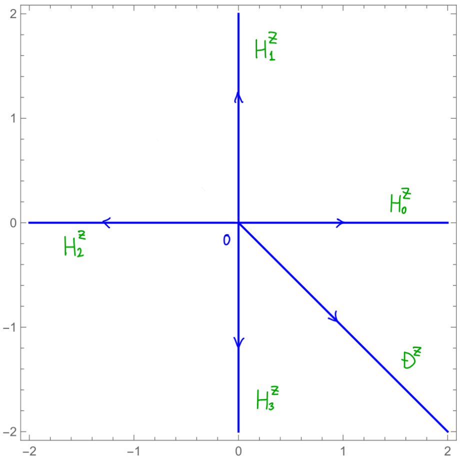


FIGURE 6. The contour Γ_Z, which consists of coordinate axes and the ray $\arg(\zeta) = -\pi/4$, and the jump matrices $G_Z(\zeta)$ for RHP 8.

This is the same setup as described in [IMY25, RHP 7] (and many other papers). RHP 8 can be solved explicitly, see [IMY25, page 21]. Indeed, let $\arg(\zeta) \in (-\frac{\pi}{4}, \frac{7\pi}{4})$ and set

$$\begin{aligned} &\Omega_0 = \left\{\zeta \in \mathbb{C} \,|\, \arg(\zeta) \in \left(-\tfrac{\pi}{4}, 0\right)\right\}, && \Omega_1 = \left\{\zeta \in \mathbb{C} \,|\, \arg(\zeta) \in \left(0, \tfrac{\pi}{2}\right)\right\}, \\ &\Omega_2 = \left\{\zeta \in \mathbb{C} \,|\, \arg(\zeta) \in \left(\tfrac{\pi}{2}, \pi\right)\right\}, && \Omega_3 = \left\{\zeta \in \mathbb{C} \,|\, \arg(\zeta) \in \left(\pi, \tfrac{3\pi}{2}\right)\right\}, \\ &\Omega_4 = \left\{\zeta \in \mathbb{C} \,|\, \arg(\zeta) \in \left(\tfrac{3\pi}{2}, \tfrac{7\pi}{4}\right)\right\}. \end{aligned}$$

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[4] Notice that we can equivalently write $s_1 = -\hat{\delta}(1)^{-2}(1 + e^{-\pi i\nu})e^{-2it}$ and $s_2 = \hat{\delta}(1)^2(1 - e^{-\pi i\nu})e^{-2it}$.

Then, the solution of RHP 8 is given by

$$Z(\zeta)e^{\frac{1}{4}\zeta^2\sigma_3} = \begin{cases} \begin{pmatrix} D_{-\nu}\left(\zeta e^{\pi \mathrm{i}/2}\right)e^{\pi\mathrm{i}\nu/2} & -\hat{\alpha}D_{\nu-1}(\zeta) \\ \mathrm{i}\hat{\beta}D_{-\nu-1}\left(\zeta e^{\pi\mathrm{i}/2}\right)e^{\pi\mathrm{i}\nu/2} & D_\nu(\zeta) \end{pmatrix}, & \zeta\in\Omega_0, \\ \begin{pmatrix} D_{-\nu}\left(\zeta e^{-\pi \mathrm{i}/2}\right)e^{-\pi\mathrm{i}\nu/2} & -\hat{\alpha}D_{\nu-1}(\zeta) \\ -\mathrm{i}\hat{\beta}D_{-\nu-1}\left(\zeta e^{-\pi\mathrm{i}/2}\right)e^{-\pi\mathrm{i}\nu/2} & D_\nu(\zeta) \end{pmatrix}, & \zeta\in\Omega_1, \\ \begin{pmatrix} D_{-\nu}\left(\zeta e^{-\pi \mathrm{i}/2}\right)e^{-\pi\mathrm{i}\nu/2} & \hat{\alpha}D_{\nu-1}\left(\zeta e^{-\pi\mathrm{i}}\right)e^{\pi\mathrm{i}\nu} \\ -\mathrm{i}\hat{\beta}D_{-\nu-1}\left(\zeta e^{-\pi\mathrm{i}/2}\right)e^{-\pi\mathrm{i}\nu/2} & D_\nu\left(\zeta e^{-\pi\mathrm{i}}\right)e^{\pi\mathrm{i}\nu} \end{pmatrix}, & \zeta\in\Omega_2, \\ \begin{pmatrix} D_{-\nu}\left(\zeta e^{-3\pi \mathrm{i}/2}\right)e^{-3\pi\mathrm{i}\nu/2} & \hat{\alpha}D_{\nu-1}\left(\zeta e^{-\pi\mathrm{i}}\right)e^{\pi\mathrm{i}\nu} \\ \mathrm{i}\hat{\beta}D_{-\nu-1}\left(\zeta e^{-3\pi\mathrm{i}/2}\right)e^{-3\pi\mathrm{i}\nu/2} & D_\nu\left(\zeta e^{-\pi\mathrm{i}}\right)e^{\pi\mathrm{i}\nu} \end{pmatrix}, & \zeta\in\Omega_3, \\ \begin{pmatrix} D_{-\nu}\left(\zeta e^{-3\pi \mathrm{i}/2}\right)e^{-3\pi\mathrm{i}\nu/2} & -\hat{\alpha}D_{\nu-1}\left(\zeta e^{-2\pi\mathrm{i}}\right)e^{2\pi\mathrm{i}\nu} \\ \mathrm{i}\hat{\beta}D_{-\nu-1}\left(\zeta e^{-3\pi\mathrm{i}/2}\right)e^{-3\pi\mathrm{i}\nu/2} & D_\nu\left(\zeta e^{-2\pi\mathrm{i}}\right)e^{2\pi\mathrm{i}\nu} \end{pmatrix}, & \zeta\in\Omega_4. \end{cases} \tag{4.8}$$

Thus, one can refine the asymptotics of $Z(\zeta)$ as $\zeta\to\infty$ as

$$Z(\zeta) = \left(I + \frac{1}{\zeta}\begin{pmatrix} 0 & -\hat{\alpha} \\ \hat{\beta} & 0 \end{pmatrix} + \frac{1}{\zeta^2}\begin{pmatrix} \frac{1}{2}\nu(1+\nu) & 0 \\ 0 & \frac{1}{2}\nu(1-\nu) \end{pmatrix} + O\left(\frac{1}{\zeta^3}\right)\right)\zeta^{-\nu\sigma_3}, \tag{4.9}$$

where $\zeta^\nu > 0$ when $\zeta > 0$ and $\arg(\zeta)\in(-\pi/4, 7\pi/4)$,

$$\hat{\alpha} = -e^{-2\pi\mathrm{i}\nu}\frac{\mathrm{i}}{s_2}\frac{\sqrt{2\pi}}{\Gamma(\nu)} \quad\text{and}\quad \hat{\beta} = -\frac{\nu}{\hat{\alpha}}. \tag{4.10}$$

Since ν is purely imaginary, the entries of $Z(\zeta)$ are entire functions, and (4.9) holds uniformly for all $|\zeta|$ large, we have that

$$|[Z_-(\zeta)]_{ij}| \leqslant M, \quad \zeta\in\mathbb{R}\cup\mathrm{i}\mathbb{R}, \tag{4.11}$$

for some constant M and all $i,j\in\{1,2\}$ (this can be independently verified by using [DLMF, Equation (12.2.5), (12.5.3), and (12.5.5)]).

4.3 Approximate Local Parametrices at $w = \pm 1$

As we indicated before, we use the following conformal map to set up a correspondence between w and ζ planes:

$$\zeta(w) = e^{\pi\mathrm{i}/2}\sqrt{\varphi(w) - 2\mathrm{i}} = e^{3\pi\mathrm{i}/4}\frac{w-1}{\sqrt{w}}, \quad w\in\mathbb{D}_\epsilon(1), \tag{4.12}$$

where $\sqrt{w}$ is the principal branch and the other root is defined so that the second equality holds. Notice that the interval $(1-\epsilon, 1)$ is mapped by $\zeta(w)$ into the ray $\arg(\zeta) = -\pi/4$. Since we had quite a bit of freedom in choosing the arcs $\Gamma^\pm_{ext}$ and $\Gamma^\pm_{int}$, we fix them around 1 so that ζ maps the arcs Γ_i, see Figure 3, into the coordinate axes. Then, the map $\sqrt{2t}\zeta(w)$ maps the jump contour on Figure 5 into Γ_Z from Figure 6 for any $t > 0$ so that the matrix $\hat{H}_i$ corresponds to the matrix H^Z_i for any $i\in\{0,1,2,3\}$. That is, the matrix

$$Z\left(\sqrt{2t}\zeta(w)\right)$$

has the same jumps as $\hat{Y}(w,t)$ except for $\hat{\delta}(w)$ and $h(w)$ being replaced by $\hat{\delta}(1)$ and $h(1)$, respectively. Now, by undoing the transformation (4.6) and (4.1) we arrive at an approximate local parametrix

$$P^{(1)}(w,t) = V^{(1)}(w,t)Z\left(\sqrt{2t}\zeta(w)\right)(w-1)^{\nu\sigma_3}B, \quad w\in\mathbb{D}_\epsilon(1), \tag{4.13}$$

where $V^{(1)}(w,t)$ could be any matrix function holomorphic in $\mathbb{D}_\epsilon(1)$. We choose this holomorphic prefactor to ensure good matching with the global parametrix and set

$$V^{(1)}(w,t) = A(w)\big(\sqrt{2t}\zeta(w)\big)^{\nu\sigma_3}(w-1)^{-\nu\sigma_3} \\ = A(w)\big(\sqrt{2t}e^{3\pi \mathrm{i}/4}w^{-1/2}\big)^{\nu\sigma_3}, \tag{4.14}$$

where $A(w)$ was defined in (3.2) and $w^{\nu/2}$ is the principal branch.

Due to (2.21), we define the local parametrix $P^{(-1)}(w)$ near $w=-1$ by setting

$$P^{(-1)}(w,t) = \sigma_1 P^{(1)}(-w,t)\sigma_1 \begin{cases} 1, & w \in \operatorname{ext}(\Gamma_{gl}), \\ -1, & w \in \operatorname{int}(\Gamma_{gl}). \end{cases} \tag{4.15}$$

If we denote the jump matrices for $P^{(\pm 1)}(w)$ by $G_{P^{(\pm 1)}}(w,t)$, then it holds that

$$G_{P^{(-1)}}(w,t) = \sigma_1 G_{P^{(1)}}(-w,t)\sigma_1. \tag{4.16}$$

4.4 Comparison between RHP 6 and Approximate Local Parametrices

For future use we now compare $Y(w,t)$ to the local paramatrices $P^{(\pm 1)}(w)$ in $\mathbb{D}_\epsilon(\pm 1)$. These matrices have matching jumps on $\Gamma_{gl} \cap \mathbb{D}_\epsilon(1)$ while

$$G_Y(w,t) = \begin{cases} \begin{pmatrix} 1 & \hat\delta(w)^2(w-1)^{2\nu}e^{2\pi \mathrm{i}\nu}\frac{h(w)-1}{h(w)}e^{t\varphi(w)} \\ 0 & 1 \end{pmatrix}, & w \in \Gamma_0, \\ \begin{pmatrix} 1 & \frac{-1}{\hat\delta(w)^2}(w-1)^{-2\nu}\frac{h(w)+1}{h(w)}e^{-t\varphi(w)} \\ 0 & 1 \end{pmatrix}, & w \in \Gamma_1, \\ \begin{pmatrix} 1 & 0 \\ \hat\delta(w)^2(w-1)^{2\nu}\frac{h(w)-1}{h(w)}e^{t\varphi(w)} & 1 \end{pmatrix}, & w \in \Gamma_2, \\ \begin{pmatrix} 1 & 0 \\ \frac{-1}{\hat\delta(w)^2}(w-1)^{-2\nu}e^{2\pi \mathrm{i}\nu}\frac{h(w)+1}{h(w)}e^{-t\varphi(w)} & 1 \end{pmatrix}, & w \in \Gamma_3, \end{cases}$$

by (4.2), (4.4), and the definition of B in (3.2). Since the jump matrix for $P^{(1)}(w)$ is obtained from $G_Y(w,t)$ by replacing $\hat\delta(w), h(w)$ with $\hat\delta(1), h(1)$ we have that

$$\begin{aligned} &G_Y(w,t)G_{P^{(1)}}(w,t)^{-1} \\ &= \begin{cases} \begin{pmatrix} 1 & (w-1)^{2\nu}e^{2\pi \mathrm{i}\nu}\left(\hat\delta(w)^2\frac{h(w)-1}{h(w)} - \hat\delta(1)^2\frac{h(1)-1}{h(1)}\right)e^{t\varphi(w)} \\ 0 & 1 \end{pmatrix}, & w \in \Gamma_0, \\ \begin{pmatrix} 1 & (w-1)^{-2\nu}\left(\frac{1}{\hat\delta(1)^2}\frac{h(1)+1}{h(1)} - \frac{1}{\hat\delta(w)^2}\frac{h(w)+1}{h(w)}\right)e^{-t\varphi(w)} \\ 0 & 1 \end{pmatrix}, & w \in \Gamma_1, \\ \begin{pmatrix} 1 & 0 \\ (w-1)^{2\nu}\left(\hat\delta(w)^2\frac{h(w)-1}{h(w)} - \hat\delta(1)^2\frac{h(1)-1}{h(1)}\right)e^{t\varphi(w)} & 1 \end{pmatrix}, & w \in \Gamma_2, \\ \begin{pmatrix} 1 & 0 \\ (w-1)^{-2\nu}e^{2\pi \mathrm{i}\nu}\left(\frac{1}{\hat\delta(1)^2}\frac{h(1)+1}{h(1)} - \frac{1}{\hat\delta(w)^2}\frac{h(w)+1}{h(w)}\right)e^{-t\varphi(w)} & 1 \end{pmatrix}, & w \in \Gamma_3. \end{cases} \end{aligned} \tag{4.17}$$

In another connection, we get on $\partial\mathbb{D}_\epsilon(1)$ that

$$A(w)^{-1}P^{(1)}(w,t)P^{(gl)}(w)^{-1}A(w) = I + \frac{1}{\sqrt{2t}\zeta(w)}\begin{pmatrix} 0 & -\tilde{\alpha}(w) \\ \tilde{\beta}(w) & 0 \end{pmatrix} + \frac{1}{2t\zeta(w)^2}\begin{pmatrix} \frac{1}{2}\nu(1+\nu) & 0 \\ 0 & \frac{1}{2}\nu(1-\nu) \end{pmatrix} + O\big(t^{-3/2}\big) \tag{4.18}$$

by (3.1), (4.13), (4.14), and (4.9) where

$$\tilde{\alpha}(w) := \big(2te^{3\pi\mathrm{i}/2}/w\big)^{\nu}\hat{\alpha}, \quad \tilde{\beta}(w) := \big(2te^{3\pi\mathrm{i}/2}/w\big)^{-\nu}\hat{\beta}. \tag{4.19}$$

In particular, we get from the definition of $A(w)$ in (3.2) that

$$\Big[P^{(1)}(w,t)P^{(gl)}(w)^{-1}\Big]_{11} = 1 + \frac{1}{\sqrt{2t}\zeta(w)}\left(\frac{w\tilde{\beta}(w)}{(w+\mathrm{i})^2} - \frac{w\tilde{\alpha}(w)}{(w-\mathrm{i})^2}\right) + \frac{1}{2t\zeta(w)^2}\frac{\nu}{2}\frac{w^2(1+\nu)+(1-\nu)}{w^2+1} + O\big(t^{-3/2}\big), \tag{4.20}$$

and

$$\Big[P^{(1)}(w,t)P^{(gl)}(w)^{-1}\Big]_{22} = 1 - \frac{1}{\sqrt{2t}\zeta(w)}\left(\frac{w\tilde{\beta}(w)}{(w+\mathrm{i})^2} - \frac{w\tilde{\alpha}(w)}{(w-\mathrm{i})^2}\right) + \frac{1}{2t\zeta(w)^2}\frac{\nu}{2}\frac{w^2(1-\nu)+(1+\nu)}{w^2+1} + O\big(t^{-3/2}\big). \tag{4.21}$$

Clearly, analogous results in $\mathbb{D}_\epsilon(-1)$ can be deduced from the symmetry relations (2.21) and (3.3) as well as the definition (4.15).

5 Small Norm Problem

We look for the solution of RHP 6 in the form

$$Y(w) = R(w)\begin{cases} P^{(gl)}(w), & w \in \mathbb{C}\backslash(\Gamma_{gl}\cup\mathbb{D}_\epsilon(1)\cup\mathbb{D}_\epsilon(-1)), \\ P^{(\pm1)}(w), & w \in \mathbb{D}_\epsilon(\pm1). \end{cases} \tag{5.1}$$

Then, the error function $R(w)$ must solve the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 9. *Find a 2×2 matrix function $R(w,t)$ such that*

1. *$R(w)$ is analytic in $\mathbb{C}\backslash\Gamma_R$, where $\Gamma_R = \Gamma_R^{out}\cup\Gamma_R^{bnd}\cup\Gamma_R^{in}$, see Figure 7, with*
$$\begin{aligned} \Gamma_R^{out} &:= \Gamma\backslash\big(\Gamma_{gl}\cup\mathbb{D}_\epsilon(1)\cup\mathbb{D}_\epsilon(-1)\big), \\ \Gamma_R^{bnd} &:= \partial\mathbb{D}_\epsilon(1)\cup\partial\mathbb{D}_\epsilon(-1), \\ \Gamma_R^{in} &:= \big((\Gamma\backslash\Gamma_{gl})\cap\mathbb{D}_\epsilon(1)\big)\cup\big((\Gamma\backslash\Gamma_{gl})\cap\mathbb{D}_\epsilon(-1)\big); \end{aligned}$$
2. *one-sided traces $R_\pm(w)$ exist a.e. on Γ_R, are bounded there, and satisfy*
$$R_+(w,t) = R_-(w,t)G_R(w,t), \quad w\in\Gamma_R,$$
where
$$G_R(w,t) = \begin{cases} P^{(gl)}(w)G_Y(w,t)P^{(gl)}(w)^{-1}, & w\in\Gamma_R^{out}, \\ P_-^{(\pm1)}(w,t)G_Y(w,t)G_{P^{(\pm1)}}(w,t)^{-1}P_-^{(\pm1)}(w,t)^{-1}, & w\in\Gamma_R^{in}, \\ P^{(\pm1)}(w,t)P^{(gl)}(w)^{-1}, & w\in\Gamma_R^{bnd}; \end{cases}$$
3. *it holds that $R(w) = I + w^{-1}R_1(t) + O\left(w^{-2}\right)$ as $w\to\infty$.*

In what follows from (2.20), (3.3), (4.15), and (4.16) that

$$E(w) = \sigma_1 E(-w)\sigma_1, \quad E(w) := G_R(w) - I. \tag{5.2}$$

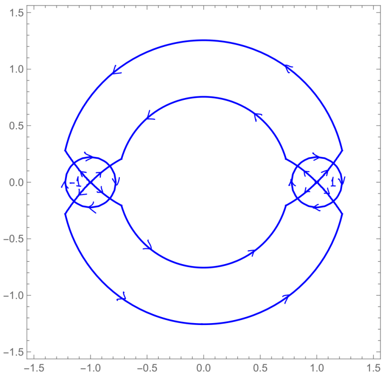

FIGURE 7. The contour Γ_R for RHP 9.

5.1 Integral Representation

Since $\hat{\delta}(w)$, $h(w)$, and $(w-1)^{\nu}$ are fixed analytic functions, and so are the entries of $P^{(gl)}(w)$, it follows from (2.8), (2.9), and (2.19) that

$$\|E\|_{L^\infty(\Gamma_R^{out})} = O(e^{-c_\epsilon t}), \tag{5.3}$$

where c_ϵ is some constant dependent on the radius ϵ of the chosen disks around $w = \pm 1$. On Γ_R^{bnd}, we have from (4.15) and (4.18) that

$$\|E\|_{L^\infty(\Gamma_R^{bnd})} = O\big(t^{-1/2}\big).$$

Next, due to the choice of the arcs $\Gamma_{int}^{\pm}$ and $\Gamma_{ext}^{\pm}$ made right after (4.12) and since $\hat{\delta}(w)$ and $h(w)$ are differentiable functions on those arcs, see (4.5), we see that off-diagonal entries of the matrices in (4.17) are of the form

$$\{\text{bounded function}\} \times \zeta e^{-t\zeta^2}$$

for $\zeta > 0$. Hence, we have that

$$E(w) = \{\text{bounded function}\} \times \zeta(w) e^{-t\zeta^2(w)} P_-^{(1)}(w) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} P_-^{(1)}(w)^{-1}$$

for $w \in \Gamma_0$ and $w \in \Gamma_1$. Clearly, similar expressions can be written for $w \in \Gamma_2$ and $w \in \Gamma_3$. Thus, we can see from (4.11), (4.13), (4.14), and (4.15) that

$$E(w) = \zeta(w) e^{-t\zeta^2(w)} O(1), \quad w \in \Gamma_R^{in},$$

where the entries of $O(1)$ are bounded functions. Since $1/\zeta'(w)$ is a bounded function in $\mathbb{D}_\epsilon(1)$ as well, a trivial calculus computation shows that

$$\|E\|_{L^p(\Gamma_R^{in})} = O\big(t^{-1/2-1/(2p)}\big) \tag{5.4}$$

for any $p \in [1, \infty]$, where the estimate in $\mathbb{D}_\epsilon(-1)$ follows from symmetry (5.2). Altogether, we get that

$$\|E\|_{L^\infty(\Gamma_R)} = O\big(t^{-1/2}\big) \quad \text{as} \quad t \to \infty. \tag{5.5}$$

Hence, by the small norm theorem, see for example [FIKN06, Theorem 8.1], there exists a unique solution of RHP 9 for all t large and

$$R(w) = I + \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_R}\frac{\rho(w')E(w')}{w'-w}dw', \quad w\notin\Gamma_R, \tag{5.6}$$

where $\rho(w') = R_-(w')$ for $w'\in\Gamma_R$. Let $(\mathcal{C}_E F)(w) := (\mathcal{C}(FE))_-(w)$, $w\in\Gamma_R$, where $\mathcal{C}$ is the Cauchy operator on Γ_R. Then,

$$\rho(w) = I + (\mathcal{C}_E\rho)(w), \quad w\in\Gamma_R,$$

which is a singular integral equation obtained by taking the traces of both sides of (5.6) on the left-hand side of Γ_R. By (5.5), $\|\mathcal{C}_E\| = O(t^{-1/2})$ and therefore it holds that

$$\rho = (\mathcal{I}-\mathcal{C}_E)^{-1}(I) = \sum_{m=0}^{\infty}\mathcal{C}_E^m I.$$

Thus, (5.6) can be written as

$$R(w) = I + \sum_{m=0}^{\infty}\mathcal{C}\left(\left(\mathcal{C}_E^m I\right)E\right)(w) =: \sum_{k=0}^{\infty}\mathcal{S}^{(k)}(w), \tag{5.7}$$

where $\mathcal{S}^{(0)}(w) = I$ and $\mathcal{S}^{(k+1)}(w) = \mathcal{C}\big(\mathcal{S}_-^{(k)}E\big)(w)$ for $k\geqslant 0$. In particular,

$$[R_1(t)]_{11} = \lim_{w\to\infty} w\big[R(w)-I\big]_{11} = \sum_{k=1}^{\infty}\lim_{w\to\infty} w\big[\mathcal{S}^{(k)}(w)\big]_{11}, \tag{5.8}$$

where we used (5.7) for the last equality. Our goal is to asymptotically evaluate the above quantity as $t\to\infty$.

5.2 Asymptotic Analysis of $\mathcal{S}^{(1)}$

By its very definition, we have that

$$\lim_{w\to\infty} w\big[\mathcal{S}^{(1)}(w)\big]_{11} = -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_R}[E(w')]_{11}dw = I_{out}+I_{bnd}+I_{in}, \tag{5.9}$$

where the summands on the right-hand side above represent integrals over Γ_R^{out}, Γ_R^{bnd}, and Γ_R^{in}, respectively. Due to (5.3), we have that

$$I_{out} = -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_R^{out}}[E(w')]_{11}dw = O(e^{-c_\epsilon t}). \tag{5.10}$$

Now, it follows from (5.2) that

$$I_{bnd} = -\frac{1}{2\pi\mathrm{i}}\int_{\partial\mathbb{D}_\epsilon(1)}\big([E(w)]_{11}-[E(w)]_{22}\big)dw.$$

According to (4.20) and (4.21) the integrand above is equal to

$$\frac{\sqrt{2}}{\sqrt{t}\zeta(w)}\left(\frac{w\tilde\beta(w)}{(w+\mathrm{i})^2}-\frac{w\tilde\alpha(w)}{(w-\mathrm{i})^2}\right)+\frac{\nu^2}{2t\zeta(w)^2}\frac{w^2-1}{w^2+1}+O\big(t^{-3/2}\big).$$

Recall that both circles $\partial\mathbb{D}_\epsilon(\pm1)$ are oriented clockwise. Recall also (4.12) and (4.19). Thus, by the Cauchy integral formula, one has that

$$\begin{aligned} I_{bnd} &= e^{3\pi\mathrm{i}/4}\frac{\tilde\alpha(1)+\tilde\beta(1)}{\sqrt{2t}}+\frac{\mathrm{i}\nu^2}{2t}+O\big(t^{-3/2}\big) \\ &= d_1e^{-2\mathrm{i}t+\nu\ln t}t^{-1/2}+d_2e^{2\mathrm{i}t-\nu\ln t}t^{-1/2}+\frac{\mathrm{i}\nu^2}{2t}+O\big(t^{-3/2}\big), \end{aligned} \tag{5.11}$$

where, in view of (4.19), (4.10), and (4.7),

$$d_1 = e^{\pi \mathrm{i}(1-2\nu)/4} \frac{2^\nu}{\hat\delta(1)^2} \frac{h(1)}{h(1)-1} \frac{\sqrt{\pi}}{\Gamma(\nu)} \quad \text{and} \quad d_2 = \frac{\mathrm{i}\nu}{2d_1}.$$

Next, we deduce from (5.2) that

$$I_{in} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_R^{in} \cap \mathbb{D}_\epsilon(1)} \big([E(w)]_{22} - [E(w)]_{11} \big) dw. \tag{5.12}$$

To compute this integral, one needs an explicit expression of

$$G_R(w,t) = P_-^{(1)}(w,t) G_Y(w,t) G_{P^{(1)}}^{-1}(w,t) P_-^{(1)}(w,t)^{-1}$$

for $w \in \Gamma_R^{in} \cap \mathbb{D}_\epsilon(1)$. We need to introduce more notation. Recall (4.8). We shall write

$$Z_-\left(\sqrt{2t}\zeta(w)\right) = \begin{pmatrix} f_1^{(i)}\left(\sqrt{2t}\zeta(w)\right) & \hat\alpha f_2^{(i)}\left(\sqrt{2t}\zeta(w)\right) \\ \hat\beta f_3^{(i)}\left(\sqrt{2t}\zeta(w)\right) & f_4^{(i)}\left(\sqrt{2t}\zeta(w)\right) \end{pmatrix}, \quad w \in \Gamma_i,$$

for $i \in \{0,1,2,3\}$. Further, we write

$$P_-^{(1)}(w,t) =: \begin{pmatrix} a^{(i)}(w,t) & b^{(i)}(w,t) \\ c^{(i)}(w,t) & d^{(i)}(w,t) \end{pmatrix}, \quad w \in \Gamma_i,$$

again, for $i \in \{0,1,2,3\}$. Then, according to (3.2), (4.13), and (4.14) we have that

$$a^{(i)}(w,t) = \begin{cases} \dfrac{\hat\alpha w}{w-\mathrm{i}} \dfrac{(*)^\nu}{(w-1)^{2\nu}} f_2^{(i)}(*) + \dfrac{1}{w+\mathrm{i}} (*)^{-\nu} f_4^{(i)}(*), & i \in \{0,3\}, \\ \dfrac{w}{w-\mathrm{i}} (*)^\nu f_1^{(i)}(*) + \dfrac{\hat\beta}{w+\mathrm{i}} \dfrac{(w-1)^{2\nu}}{(*)^\nu} f_3^{(i)}(*), & i \in \{1,2\}, \end{cases}$$

where, for brevity, $* = \sqrt{2t}\zeta(w)$,

$$b^{(i)}(w,t) = \begin{cases} \dfrac{-w}{w-\mathrm{i}} (*)^\nu f_1^{(i)}(*) - \dfrac{\hat\beta}{w+\mathrm{i}} \dfrac{(w-1)^{2\nu}}{(*)^\nu} f_3^{(i)}(*), & i \in \{0,3\}, \\ \dfrac{\hat\alpha w}{w-\mathrm{i}} \dfrac{(*)^\nu}{(w-1)^{2\nu}} f_2^{(i)}(*) + \dfrac{1}{w+\mathrm{i}} (*)^{-\nu} f_4^{(i)}(*), & i \in \{1,2\}, \end{cases}$$

and

$$c^{(i)}(w,t) = \begin{cases} \dfrac{-\hat\alpha}{w-\mathrm{i}} \dfrac{(*)^\nu}{(w-1)^{2\nu}} f_2^{(i)}(*) + \dfrac{w}{w+\mathrm{i}} (*)^{-\nu} f_4^{(i)}(*), & i \in \{0,3\}, \\ \dfrac{-1}{w-\mathrm{i}} (*)^\nu f_1^{(i)}(*) + \dfrac{\hat\beta w}{w+\mathrm{i}} \dfrac{(w-1)^{2\nu}}{(*)^\nu} f_3^{(i)}(*), & i \in \{1,2\}, \end{cases}$$

while

$$d^{(i)}(w,t) = \begin{cases} \dfrac{1}{w-\mathrm{i}} (*)^\nu f_1^{(i)}(*) - \dfrac{\hat\beta w}{w+\mathrm{i}} \dfrac{(w-1)^{2\nu}}{(*)^\nu} f_3^{(i)}(*), & i \in \{0,3\}, \\ \dfrac{-\hat\alpha}{w-\mathrm{i}} \dfrac{(*)^\nu}{(w-1)^{2\nu}} f_2^{(i)}(*) + \dfrac{w}{w+\mathrm{i}} (*)^{-\nu} f_4^{(i)}(*), & i \in \{1,2\}. \end{cases}$$

Denote the off-diagonal entry of $G_Y(w,t)G_{P^{(1)}}^{-1}(w,t)$ on Γ_i by $g^{(i)}(w,t)$, see (4.17). Then,

$$g^{(0)}(w,t) = e^{2\pi \mathrm{i}\nu} g^{(2)}(w,t) = (w-1)^{2\nu} e^{2\pi \mathrm{i}\nu} \Big(\hat\delta(w)^2 \tfrac{h(w)-1}{h(w)} - \hat\delta(1)^2 \tfrac{h(1)-1}{h(1)} \Big) e^{t\varphi(w)},$$
$$g^{(3)}(w,t) = e^{2\pi \mathrm{i}\nu} g^{(1)}(w,t) = (w-1)^{-2\nu} e^{2\pi \mathrm{i}\nu} \Big(\tfrac{1}{\hat\delta(1)^2} \tfrac{h(1)+1}{h(1)} - \tfrac{1}{\hat\delta(w)^2} \tfrac{h(w)+1}{h(w)} \Big) e^{-t\varphi(w)}.$$

Recall that all our matrices have determinant 1. Thus, we have

$$[E(w)]_{22} - [E(w)]_{11} = 2g^{(i)}(w,t) \begin{cases} a^{(i)}(w,t)c^{(i)}(w,t), & i \in \{0,1\}, \\ -b^{(i)}(w,t)d^{(i)}(w,t), & i \in \{2,3\}. \end{cases}$$

If we set

$$\ell^{(i)}(w,t) := \begin{cases} a^{(0)}(w,t)c^{(0)}(w,t), & i=0, \\ a^{(1)}(w,t)c^{(1)}(w,t), & i=1, \\ b^{(2)}(w,t)d^{(2)}(w,t), & i=2, \\ b^{(3)}(w,t)d^{(3)}(w,t), & i=3, \end{cases}$$

then it holds that

$$\ell^{(i)}(w,t) = -\hat{\alpha}^2\frac{w}{(w-\mathrm{i})^2}\frac{(*)^{2\nu}f_2^{(i)}(*)^2}{(w-1)^{4\nu}} + \hat{\alpha}\frac{w^2-1}{w^2+1}\frac{f_2^{(i)}(*)f_4^{(i)}(*)}{(w-1)^{2\nu}} + \frac{w}{(w+\mathrm{i})^2}\frac{f_4^{(i)}(*)^2}{(*)^{2\nu}} \tag{5.13}$$

when $i \in \{0,2\}$, where again $* = \sqrt{2t}\zeta(w)$, and

$$\begin{aligned} \ell^{(i)}(w,t) = -\frac{w}{(w-\mathrm{i})^2}(*)^{2\nu}f_1^{(i)}(*)^2 + \hat{\beta}\frac{w^2-1}{w^2+1}(w-1)^{2\nu}f_1^{(i)}(*)f_3^{(i)}(*) \\ + \hat{\beta}^2\frac{w}{(w+\mathrm{i})^2}\frac{(w-1)^{4\nu}}{(*)^{2\nu}}f_3^{(i)}(*)^2 \end{aligned}$$

when $i \in \{1,3\}$. We can now rewrite (5.12) as

$$\begin{aligned} I_{in} = \frac{1}{\pi\mathrm{i}}\int_{\Gamma_0} g^{(0)}(w,t)\ell^{(0)}(w,t)dw - \frac{1}{\pi\mathrm{i}}\int_{\Gamma_2} g^{(2)}(w,t)\ell^{(2)}(w,t)dw \\ + \frac{1}{\pi\mathrm{i}}\int_{\Gamma_1} g^{(1)}(w,t)\ell^{(1)}(w,t)dw - \frac{1}{\pi\mathrm{i}}\int_{\Gamma_3} g^{(3)}(w,t)\ell^{(3)}(w,t)dw. \end{aligned} \tag{5.14}$$

We continue by analyzing the first integral in (5.14). The product of $g^{(0)}(w,t)$ and the middle summand in (5.13) can be written as

$$2\mathrm{i}\hat{\alpha}e^{2\pi\mathrm{i}\nu+2\mathrm{i}t}f_2^{(0)}\big(\sqrt{2t}\zeta(w)\big)f_4^{(0)}\big(\sqrt{2t}\zeta(w)\big)\zeta(w)\zeta'(w)F(w)e^{-t\zeta^2(w)},$$

where

$$F(w) = \frac{w^2}{w^2+1}\left(\hat{\delta}(w)^{2\frac{h(w)-1}{h(w)}} - \hat{\delta}(1)^{2\frac{h(1)-1}{h(1)}}\right).$$

Observe that $F(w)$ is a differentiable function on Γ_0 that vanishes at $w=1$. Recall that $\zeta(w)$ maps Γ_0 into the positive reals. Then, the integral of interest can be written as

$$\begin{aligned} I_{in}^{(0,2)} &:= \frac{2\hat{\alpha}}{\pi}e^{2\pi\mathrm{i}\nu+2\mathrm{i}t}\int_0^{\zeta_0} f_2^{(0)}\big(\sqrt{2t}\zeta\big)f_4^{(0)}\big(\sqrt{2t}\zeta\big)\zeta F(w(\zeta))e^{-t\zeta^2}d\zeta \\ &= \frac{\hat{\alpha}}{\pi t}e^{2\pi\mathrm{i}\nu+2\mathrm{i}t}\int_0^{\sqrt{2t}\zeta_0} f_2^{(0)}(s)f_4^{(0)}(s)F\big(w(s/\sqrt{2t})\big)se^{-s^2/2}ds, \end{aligned}$$

where ζ_0 is the value of $\zeta(w)$ at the point where Γ_0 meets $\partial\mathbb{D}_\epsilon(1)$. Observe that $\hat{\alpha}e^{2\mathrm{i}t}$ is in fact independent of t, see (4.10) and (4.7). Recall that the functions $f_2^{(0)}(s)$ and $f_4^{(0)}(s)$ are bounded on the positive reals, see (4.11). Then,

$$\big|I_{in}^{(0,2)}\big| \leqslant Mt^{-3/2}\int_0^{\sqrt{2t}\zeta_0} s^2e^{-s^2/2}ds = O\big(t^{-3/2}\big) \quad \text{as} \quad t\to\infty \tag{5.15}$$

for some constant $M>0$, where we used estimates $|F(w)| \leqslant c|w-1|$ and $|w(\zeta)| \leqslant c|\zeta|$ for some constant c (both functions are continuously differentiable).

Next, we consider the product of $g^{(0)}(w,t)$ and the last summand in (5.13). Since this last term does not vanish at 1, our analysis needs additional steps. This product can be written as

$$2\frac{e^{\pi\mathrm{i}\nu/2-3\pi\mathrm{i}/4+2\mathrm{i}t}}{(2t)^\nu}f_4^{(0)}\big(\sqrt{2t}\zeta(w)\big)^2\zeta'(w)G(w)e^{-t\zeta^2(w)}$$

(notice that we do not have $\zeta(w)$ in the above formula this time), where

$$G(w) = \frac{w^{5/2+\nu}}{(w+1)(w+\mathrm{i})^2}\left(\hat{\delta}(w)^{2\frac{h(w)-1}{h(w)}} - \hat{\delta}(1)^{2\frac{h(1)-1}{h(1)}}\right).$$

Because the first factor above is an analytic function in $\mathbb{D}_\epsilon(1)$, we can, in fact, write

$$G(w) = \left(\frac{1}{4\mathrm{i}} + \zeta(w)G_1(w)\right)\left(\hat{\delta}(w)^{2\frac{h(w)-1}{h(w)}} - \hat{\delta}(1)^{2\frac{h(1)-1}{h(1)}}\right),$$

where $G_1(w)$ is an analytic function in $\mathbb{D}_\epsilon(1)$. Similarly, as we have shown in (4.5), the second factor in the formula for $G(w)$ is a differentiable function on Γ_0 whose derivative is Hölder continuous with any index < 1. Hence,

$$G(w) = \zeta(w)\left(\frac{1}{4\mathrm{i}} + \zeta(w)G_1(w)\right)\left(e^{-3\pi\mathrm{i}/4}d + G_2(w)\right),$$

where $G_2(w)$ is a continuous function in Γ_0 such that $|G_2(w)| \leqslant c_\varepsilon |w-1|^{1-2\varepsilon}$ for any $\varepsilon > 0$, and d is the value of the derivative of $\hat{\delta}(w)^2(1-h(w)^{-1})$ at 1. Thus,

$$\begin{aligned} I_{in}^{(0,3)} &:= 2\frac{e^{\pi\mathrm{i}\nu/2+3\pi\mathrm{i}/4+2\mathrm{i}t}}{\pi(2t)^\nu}\int_0^{\zeta_0} f_4^{(0)}\big(\sqrt{2t}\zeta\big)^2 G(w(\zeta))e^{-t\zeta^2}d\zeta \\ &= d\frac{e^{\pi\mathrm{i}\nu/2+2\mathrm{i}t}}{2\pi\mathrm{i}(2t)^\nu}\int_0^{\zeta_0} \zeta f_4^{(0)}\big(\sqrt{2t}\zeta\big)^2 e^{-t\zeta^2}d\zeta + O_\varepsilon\big(t^{-3/2+\varepsilon}\big), \end{aligned}$$

where $O_\varepsilon(t^{-3/2+\varepsilon})$ is obtained as in (5.15). The integral above can be rewritten as

$$\frac{1}{2t}\int_0^{\sqrt{2t}\zeta_0} sD_\nu^2(s)ds = \frac{1}{2t}\int_0^\infty sD_\nu^2(s)ds + o\big(e^{-ct}\big) \tag{5.16}$$

for some constant $c > 0$, see (4.8). Altogether, we get for some explicit constant $C_\nu^{(0,3)}$ that

$$I_{in}^{(0,3)} = C_\nu^{(0,3)}e^{2\mathrm{i}t-\nu\ln t}t^{-1} + O_\varepsilon(t^{-3/2+\varepsilon}) \quad \text{as} \quad t\to\infty. \tag{5.17}$$

Now, we consider the product of $g^{(0)}(w,t)$ and the first summand in (5.13). This product can be written as

$$-2\hat{\alpha}^2 e^{7\pi\mathrm{i}\nu/2-3\pi\mathrm{i}/4+2\mathrm{i}t}(2t)^\nu f_2^{(0)}\big(\sqrt{2t}\zeta(w)\big)^2\zeta'(w)H(w)e^{-t\zeta^2(w)},$$

where

$$H(w) = \frac{w^{5/2-\nu}}{(w+1)(w-\mathrm{i})^2}\left(\hat{\delta}(w)^{2\frac{h(w)-1}{h(w)}} - \hat{\delta}(1)^{2\frac{h(1)-1}{h(1)}}\right).$$

Recall, see (4.10) and (4.7), that $\hat{\alpha}^2$ is equal to $e^{-4\mathrm{i}t}$ times a constant that depends only on ν. Hence, repeating the steps leading to (5.17), we get for some explicit constant $C_\nu^{(0,1)}$ that

$$\begin{aligned} I_{in}^{(0,1)} &:= \frac{2}{\pi}\hat{\alpha}^2 e^{7\pi\mathrm{i}\nu/2-\pi\mathrm{i}/4+2\mathrm{i}t}(2t)^\nu\int_0^{\zeta_0} f_2^{(0)}\big(\sqrt{2t}\zeta\big)^2 H(w(\zeta))e^{-t\zeta^2}d\zeta \\ &= C_\nu^{(0,1)}e^{-2\mathrm{i}t+\nu\ln t}t^{-1} + O_\varepsilon(t^{-3/2+\varepsilon}) \quad \text{as} \quad t\to\infty. \end{aligned} \tag{5.18}$$

By combining (5.15), (5.17), and (5.18) we see that the first integral in (5.14) behaves like

$$C_\nu^{(0,1)}e^{-2\mathrm{i}t+\nu\ln t}t^{-1} + C_\nu^{(0,3)}e^{2\mathrm{i}t-\nu\ln t}t^{-1} + O_\varepsilon(t^{-3/2+\varepsilon}) \quad \text{as} \quad t\to\infty. \tag{5.19}$$

When it comes to the second integral in (5.14) recall that $g^{(2)}(w,t) = e^{-2\pi\mathrm{i}\nu}g^{(0)}(w,t)$ while $\ell^{(0)}(w,t)$ and $\ell^{(2)}(w,t)$ differ only in the choice of the functions $f^{(i)}$. Now, we get from (4.8) that

$$f_2^{(2)}(s) = -e^{\pi\mathrm{i}\nu}f_2^{(0)}(-s) \quad \text{and} \quad f_4^{(2)}(s) = e^{\pi\mathrm{i}\nu}f_4^{(0)}(-s).$$

Since $\zeta(w)$ maps Γ_2 into the negative reals, all our previous estimates carry over unchanged with (5.16) becoming

$$\frac{1}{2t}\int_0^{-\sqrt{2t}\zeta_2} sD_\nu^2(-s)ds = \frac{1}{2t}\int_0^\infty sD_\nu^2(s)ds + o\big(e^{-ct}\big),$$

where $-\zeta_2$ is the value of $\zeta(w)$ at the point where Γ_2 meets $\partial\mathbb{D}_\epsilon(1)$. Thus, the second integral in (5.14) behaves exactly as in (5.19). Due to the minus sign in front of it, the contribution of the first two summands in (5.14) to I_{in} is of order $O_\varepsilon(t^{-3/2+\varepsilon})$.

The other two integrals can be estimated similarly. We only need to observe that $g^{(3)}(w,t) = e^{2\pi i\nu}g^{(1)}(w,t)$, $i\zeta(w)$ maps Γ_1 and Γ_3 into the negative and positive reals while

$$f_1^{(3)}(s) = e^{-\pi i\nu}f_1^{(1)}(-s) \quad \text{and} \quad f_3^{(3)}(s) = -e^{-\pi i\nu}f_3^{(1)}(-s).$$

Altogether, we get that

$$I_{in} = O_\varepsilon(t^{-3/2+\varepsilon}) \quad \text{as} \quad t\to\infty \quad \text{for any} \quad \varepsilon>0. \tag{5.20}$$

Respectively, we get from (5.9), (5.10), (5.11), and (5.20) that

$$\lim_{w\to\infty} w\big[\mathcal{S}^{(1)}(w)\big]_{11} = d_1e^{-2it+\nu\ln t}t^{-1/2} + d_2e^{2it-\nu\ln t}t^{-1/2} + \frac{i\nu^2}{2t} + O_\varepsilon\big(t^{-3/2+\varepsilon}\big). \tag{5.21}$$

5.3 Asymptotic Analysis of $\mathcal{S}^{(2)}$

Next we analyze $\mathcal{S}^{(2)}(w) = C(\mathcal{S}_-^{(1)}E)(w) = C((C_-E)E)(w)$. We have that

$$\lim_{w\to\infty} w\,\mathcal{S}^{(2)}(w) = -\frac{1}{2\pi i}\int_{\Gamma_R}\left(\lim_{s\to z\in\Gamma_{R-}}\frac{1}{2\pi i}\int_{\Gamma_R}\frac{E(w')}{w'-s}dw'\right)E(z)dz.$$

We write the integral in parentheses above as a sum of two terms, say $J_1(s)$ and $J_2(s)$, where $J_1(s)$ is obtained by integrating over Γ_R^{bnd} and $J_2(s)$ by integrating over $\Gamma_R^{out}\cup\Gamma_R^{in}$.

We start with $J_2(s)$. Observe that $J_{2-}(s) = C_-(\tilde{E})(s)$, where $\tilde{E}(w) = E(w)$ on $\Gamma_R^{out}\cup\Gamma_R^{in}$ and $\tilde{E}(w) = 0$ on Γ_R^{bnd}. Hence,

$$\|J_{2-}\|_{L^2(\Gamma_R)} \leqslant \|C_-\|\|\tilde{E}\|_{L^2(\Gamma_R)} = O\big(t^{-3/4}\big)$$

by (5.3) and (5.4). Thus, we have that

$$\begin{aligned}\|J_{2-}E\|_{L^1(\Gamma_R)} &\leqslant \|E\|_{L^\infty(\Gamma_R)}\|J_{2-}\|_{L^1(\Gamma_R)}\\ &\leqslant |\Gamma_R|^{1/2}\|E\|_{L^\infty(\Gamma_R)}\|J_{2-}\|_{L^2(\Gamma_R)} = O\big(t^{-5/4}\big),\end{aligned}$$

where we used (5.5) on the last step. That is, it holds that

$$\lim_{w\to\infty} w\big[\mathcal{S}^{(2)}(w)\big]_{11} = -\frac{1}{2\pi i}\int_{\Gamma_R}\big[J_{1-}(z)E(z)\big]_{11}dz + O\big(t^{-5/4}\big) \tag{5.22}$$

as $t\to\infty$. When it comes to $J_1(s)$, it follows from (5.2) that

$$J_1(s) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_\epsilon(1)}\frac{E(w')}{w'-s}dw' + \frac{1}{2\pi i}\int_{\partial\mathbb{D}_\epsilon(1)}\frac{\sigma_1E(w')\sigma_1}{w'+s}dw'.$$

This formula immediately yields symmetry $J_1(s) = \sigma_1J_1(-s)\sigma_1$. Denote the integral in (5.22) by I. Then,

$$I = \frac{1}{2\pi i}\int_{\Gamma_R\cap\{\mathrm{Re}(w)>0\}}\big[\sigma_1J_{1-}(w)E(w)\sigma_1 - J_{1-}(w)E(w)\big]_{11}dw,$$

where we used (5.2) and the above symmetry of $J_1(s)$. By (4.18), it holds that

$$E(w) = E_1(w) + E_2(w), \quad E_1(w) = \frac{1}{\sqrt{t}}\frac{M(w)}{w-1}, \quad E_2(w) = O\big(t^{-1}\big), \tag{5.23}$$

where $M(w)$ is an analytic matrix function around 1 given by

$$M(w) := e^{-3\pi \mathrm{i}/4}\sqrt{\frac{w}{2}}A(w)\begin{pmatrix} 0 & -\tilde{\alpha}(w) \\ \tilde{\beta}(w) & 0 \end{pmatrix}A(w)^{-1}.$$

Write $J_1(s) = J_{1,1}(s) + J_{1,2}(s)$, where this decomposition is based on the above decomposition $E(w)$. Extend $E_2(w)$ by zero to the entire contour Γ_R. Then, similarly to the case of $J_2(s)$, we have that

$$\|J_{1,2-}\|_{L^2(\Gamma_R)} = O\big(t^{-1}\big) \quad \Rightarrow \quad \|J_{1,2-}E\|_{L^1(\Gamma_R)} = O\big(t^{-3/2}\big).$$

Thus, we have that

$$I = I_1 + O\big(t^{-3/2}\big) \quad \text{as} \quad t \to \infty, \tag{5.24}$$

where I_1 is defined exactly as I, but with $J_1(w)$ replaced by $J_{1,1}(w)$. Now, $J_{1,1}(s)$ can be computed explicitly. We get from the residue theorem that

$$J_{1,1}(s) = \frac{1}{\sqrt{t}}\left(-N(s) - \frac{\sigma_1 M(1)\sigma_1}{s+1}\right), \quad s \in \mathbb{D}_\epsilon(1),$$

where $N(s) = (M(s) - M(1))/(s-1)$ (recall that $\partial\mathbb{D}_\epsilon(1)$ and $\partial\mathbb{D}_\epsilon(-1)$ are oriented clockwise). We have that

$$M(w) = e^{-3\pi \mathrm{i}/4}\sqrt{\frac{w}{2}}\begin{pmatrix} \dfrac{w\tilde{\beta}(w)}{(w+\mathrm{i})^2} - \dfrac{w\tilde{\alpha}(w)}{(w-\mathrm{i})^2} & -\dfrac{\tilde{\beta}(w)}{(w+\mathrm{i})^2} - \dfrac{w^2\tilde{\alpha}(w)}{(w-\mathrm{i})^2} \\ \dfrac{w^2\tilde{\beta}(w)}{(w+\mathrm{i})^2} + \dfrac{\tilde{\alpha}(w)}{(w-\mathrm{i})^2} & -\dfrac{w\tilde{\beta}(w)}{(w+\mathrm{i})^2} + \dfrac{w\tilde{\alpha}(w)}{(w-\mathrm{i})^2} \end{pmatrix}.$$

In particular, it holds that $\sigma_1 M(1)\sigma_1 = -M(1)$. Thus,

$$J_{1,1}(s) = \frac{1}{\sqrt{t}}\left(\frac{M(1)}{s+1} - N(s)\right), \quad s \in \mathbb{D}_\epsilon(1). \tag{5.25}$$

Similarly, we get that

$$J_{1,1}(s) = \frac{1}{\sqrt{t}}\frac{2s}{s^2-1}M(1), \quad s \notin \overline{\mathbb{D}_\epsilon(1) \cup \mathbb{D}_\epsilon(-1)}. \tag{5.26}$$

Write $I_1 = I_1^{out} + I_1^{bnd} + I_1^{in}$ depending on whether we integrate over Γ_R^{out}, Γ_R^{bnd}, or Γ_R^{in}. We immediately get from (5.3) and (5.26) that

$$I_1^{out} = O(e^{-c_\epsilon t}) \quad \text{as} \quad t \to \infty. \tag{5.27}$$

We further get from (5.4) and (5.25) that

$$I_1^{in} = O\left(t^{-1/2}\|E\|_{L^1(\Gamma_R^{in})}\right) = O\big(t^{-3/2}\big). \tag{5.28}$$

Next, it follows from (5.23), (5.25), and the relation $\sigma_1 M(1)\sigma_1 = -M(1)$ that

$$I_1^{bnd} = \frac{1}{t}\frac{1}{2\pi \mathrm{i}}\int_{\partial\mathbb{D}_\epsilon(1)}\big[N(w)M(w) - \sigma_1 N(w)M(w)\sigma_1\big]_{11}\frac{dw}{w-1} + O\big(t^{-3/2}\big).$$

We get from the residue theorem that the integral above is equal to

$$\begin{aligned}\big[\sigma_1 M'(1)M(1)\sigma_1 - M'(1)M(1)\big]_{11} &= -\big[\big(\sigma_1 M'(1)\sigma_1 + M'(1)\big)M(1)\big]_{11} \\ &= -\big([M'(1)]_{12} + [M'(1)]_{21}\big)[M(1)]_{21},\end{aligned}$$

where we used the fact that $M(w)$ is traceless. Thus,

$$\begin{aligned} I_1^{bnd} &= \frac{\mathrm{i}}{4t}\big(\tilde{\beta}(1)^2 - \tilde{\alpha}(1)^2\big) + O\big(t^{-3/2}\big) \\ &= c_1 e^{4\mathrm{i}t - 2\nu\ln t}t^{-1} + c_2 e^{-4\mathrm{i}t + 2\nu\ln t}t^{-1} + O\big(t^{-3/2}\big) \quad \text{as} \quad t \to \infty \end{aligned} \tag{5.29}$$

for some explicit constants c_1, c_2, where one needs to recall (4.7), (4.10), and (4.19). Gathering together (5.22), (5.24), (5.27), (5.28), and (5.29), we get that

$$\lim_{w\to\infty} w\big[\mathcal{S}^{(2)}(w)\big]_{11} = c_1 e^{-4\mathrm{i}t+2\nu\ln t}t^{-1} + c_2 e^{4\mathrm{i}t-2\nu\ln t}t^{-1} + O\big(t^{-5/4}\big) \quad \text{as} \quad t\to\infty. \tag{5.30}$$

5.4 Proof of Theorem 1

From the very definition of $\mathcal{S}^{(k)}(w)$ we get that

$$\Big|\lim_{w\to\infty} w\mathcal{S}^{(k)}(w)\Big| \leqslant \frac{1}{2\pi}\|\mathcal{S}_-^{(k-1)}E\|_{L^1(\Gamma_R)} \leqslant \frac{|\Gamma_R|^{1/2}}{2\pi}\|E\|_{L^\infty(\Gamma_R)}\|\mathcal{S}_-^{(k-1)}\|_{L^2(\Gamma_R)}.$$

Iterating the last estimate gives

$$\Big|\lim_{w\to\infty} w\mathcal{S}^{(k)}(w)\Big| \leqslant \frac{|\Gamma_R|^{1/2}}{2\pi}\|I\|_{L^2(\Gamma_R)}\|C_-\|^{k-1}\|E\|^k_{L^\infty(\Gamma_R)}.$$

Hence, we have that

$$\sum_{k=3}^{\infty}\Big|\lim_{w\to\infty} w\mathcal{S}^{(k)}(w)\Big| \leqslant \frac{|\Gamma_R|^{1/2}}{2\pi}\frac{\|I\|_{L^2(\Gamma_R)}}{\|C_-\|}\frac{\|C_-\|^3\|E\|^3_{L^\infty(\Gamma_R)}}{1-\|C_-\|\|E\|_{L^\infty(\Gamma_R)}} = O\big(t^{-3/2}\big)$$

as $t\to\infty$, where the last conclusion follows from (5.5). Thus, we get from (5.8), (5.21), and (5.30) that

$$\begin{aligned}[R_1(t)]_{11} = d_1 e^{-2\mathrm{i}t+\nu\ln t}t^{-1/2} + d_2 e^{2\mathrm{i}t-\nu\ln t}t^{-1/2} + c_1 e^{-4\mathrm{i}t+2\nu\ln t}t^{-1} \\ + c_2 e^{4\mathrm{i}t-2\nu\ln t}t^{-1} + \frac{\mathrm{i}\nu^2}{2t} + O\big(t^{-5/4}\big) \quad \text{as} \quad t\to\infty.\end{aligned}$$

It follows from (3.1) and (5.1) that

$$[Y_1(t)]_{11} = [R_1(t)]_{11} + \mathrm{i}.$$

Therefore, we get from (2.18) and (2.22) that

$$\sigma'(t) = t + \frac{1}{\pi}\int_{-1}^{1}\ln|\tanh\beta\zeta|d\zeta + \mathrm{i}[R_1(t)]_{11}.$$

Observe that for a given complex number u with $\mathrm{Re}(u) > 0$ and a real number x, it holds that

$$\begin{aligned}\mathrm{i}x\int_1^t s^{-u}e^{\mathrm{i}xs}ds &= t^{-u}e^{\mathrm{i}xt} - e^{\mathrm{i}x} + u\int_1^t s^{-u-1}e^{\mathrm{i}xs}ds \\ &= u\int_1^\infty s^{-u-1}e^{\mathrm{i}xs}ds - e^{\mathrm{i}x} + O\big(t^{-\mathrm{Re}(u)}\big) \quad \text{as} \quad t\to\infty\end{aligned}$$

because the last integral above is absolutely convergent. Hence, we get that

$$\sigma(t) = \frac{1}{2}t^2 + \frac{t}{\pi}\int_{-1}^{1}\ln|\tanh\beta\zeta|d\zeta - \frac{\nu^2}{2}\ln t + C + O\big(t^{-1/4}\big)$$

as $t\to\infty$ for some constant C. The claim of Theorem 1 now follows from (1.1), the definition of ν in (4.3), and the definition of $h(w)$ in (2.1).

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