ADVANCING FRONTS FOR THE THIN-FILM EQUATION WITH NULL SLIP AND REPULSIVE POTENTIALS: THE CASE OF PARTIAL WETTING

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ABSTRACT. For negative values of the spreading coefficient (that is, in the so-called "partial wetting" regime), we prove that the thin-film equation with zero slip and repulsive potentials P of the form $P(h) \approx h^{1-m}$ as $h \to 0$, $m > 1$, admits for any positive speed a one-parameter family of travelling-wave solutions with a contact line and (as in standard slippage models) a logarithmically-corrected linear behaviour as $h \to +\infty$. These waves have locally finite rate of dissipation for any $m > 1$ and locally finite energy for any $m \in (1, 3)$. The result thus confirms that mildly repulsive potentials effectively resolve the no-slip paradox. The family is parametrized by a thermodynamically consistent contact-line condition which reduces to the classical fixed microscopic contact-angle one if $P \equiv 0$.

1. Introduction and results

This manuscript is concerned with the following third order nonlinear ODE:

$$H^2 \left(H_{yy} - P'(H)\right)_y = -U. \tag{1.1a}$$

Equation (1.1a) describes travelling wave solutions to the *thin-film equation*, a PDE which models the height H of a liquid film spreading over a horizontal solid substrate in the regime of lubrication approximation [25, 28, 34, 23, 33]. We are interested in the case where a *contact line* exists, i.e., a triple junction where the liquid, the surrounding gas or vapour, and the solid substrate meet. For travelling wave solutions, this amounts to the existence of a point $y_0 \in \mathbb{R}$ where $H(y_0) = 0$. Such point may be set to be equal to zero using translation invariance of (1.1a):

$$H(0) = 0, \qquad H > 0 \ \text{ in } \mathbb{R}_+ = (0, +\infty). \tag{1.1b}$$

Pointwise solutions $H \in C^3(\mathbb{R}_+)$ to (1.1a) satisfying (1.1b) will be called *fronts* in what follows. They are advancing if $U > 0$ or receding if $U < 0$.

The nonlinearity H^2 in (1.1a) corresponds to assuming zero slip at the liquid-solid interface. If $P \equiv 0$, it is well-known that advancing fronts do not exist, whereas receding ones display an unbounded rate of energy dissipation at the contact line (see Remark 1.2 and the recent discussion in [31]). This is the manifestation of the no-slip paradox [30, 15] in lubrication approximation. After the discovery of this paradox, a number of enrichments of the basic model have been introduced, all of them accounting for "microscopic" physics (we refer to the reviews [34, 11, 5, 40] and to [14] for discussions): among them slip conditions, shear-thinning rheologies, and short-range repulsive intermolecular potentials of the form $P(H) = O(H^{1-m})$ as $H \to 0^+$, $m > 1$, which penalize closeness between the solid and the liquid-air interface, and $P(H) \to 0$ as $H \to +\infty$ (thus neglecting gravity; see [2] for a discussion of its effect). Our focus is on the latter model.

The idea that a liquid film can spread because of a gradient of the disjoining pressure $-P'$ dates back to [41, 10, 27]; in his review [11, Section IV.C.3] de Gennes presents a heuristic analysis of advancing fronts for van der Waals potentials. In fact, repulsive potentials with $m \geq 3$ have been extensively used in numerical simulations and asymptotic studies, mainly in relation to dewetting

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phenomena (see e.g. [3, 1, 35], the discussion in [14] and the review [44]). On the other hand, a rigorous study of fronts in the case of mildly repulsive potentials ($m \in (1,3)$) has not yet been performed to our knowledge. Their formal and numerical analysis has been recently carried out in [14], supporting the existence of generic families of fronts and thus the fact that such potentials may stand as an alternative resolution to the no-slip paradox.

The goal of this manuscript is to start giving rigorous bases to the aforementioned formal analysis. Our focus is on a class of advancing fronts which exhibit a logarithmically corrected linear behavior for large y:

$$\lim_{y\to+\infty} \frac{H_y^3(y)}{\ln y} = 3U, \qquad U > 0 \tag{1.1c}$$

(the limiting value may be easily inferred from a leading-order expansion of (1.1a)). These *linear-log fronts* are known to be the appropriate ones for connecting to the liquid bulk, allowing to develop matched asymptotic studies for spreading droplets [43, 25, 28, 9, 18, 26, 29, 4, 17, 16, 36, 8], parts of which were made rigourous in [22, 20, 12]. In [14] it has been conjectured that for any $U > 0$ there exists a one-parameter family of such fronts; in addition, a criterion has been proposed which is expected to select a unique linear-log front. The criterion is based on thermodynamically consistent contact-line conditions modeling friction *at* the contact line, in the spirit of [37, 38, 39] (see also [8, 21]). For travelling wave solutions, it reads as

$$\lim_{y\to 0^+} \left(\tfrac{1}{2}H_y^2 - P(H(y))\right) = \hat{\mu}_{\rm CL} U - S =: \Theta_0, \tag{1.1d}$$

where $\hat{\mu}_{\rm CL} \geq 0$ is proportional to contact-line friction and S is the (non-dimensional) *spreading coefficient*. The latter one is determined by the values of the three surface tension coefficients of the air-liquid-solid system: $S < 0$, $S = 0$ and $S > 0$ correspond to the cases of *partial wetting*, *complete wetting* and *dry (complete) wetting*, respectively (see [11]; see also [13] for a discussion of the statics in these different cases). Note that in the more classical case of null contact-line friction, $\hat{\mu}_{\rm CL} = 0$, (1.1d) coincides with the standard fixed contact-angle condition if $P \equiv 0$, and in that case necessarily $-S = \Theta_0 \geq 0$. Hence the case $S \leq 0$ is the relevant one for the contact-line models which are commonly used in the analysis of the thin-film equation with slippage.

Our main result confirms the above-mentioned formal analysis under assumptions that include in particular the case of a repulsive potential ($P' < 0$ in $\mathbb{R}_+$) in the partial wetting regime ($S < 0$). By a solution to (1.1) we mean a function $H \in C^3(\mathbb{R}_+) \cap C^0(\bar{R}_+)$ which satisfies (1.1a) pointwise and such that (1.1b)-(1.1d) hold.

Theorem 1.1. *Let $U > 0$, $\Theta_0 > 0$, and $P \in C^2(\mathbb{R}_+)$ such that*

$$P(H) = \tfrac{A}{m-1} H^{1-m}(1 + o(1)) \quad \text{as } h \to 0^+ \text{ for some } m > 1,\ A > 0, \tag{1.2a}$$

$$P(+\infty) = P'(+\infty) = 0, \quad H^{p+1}P''(H) = O(1) \quad \text{as } H \to +\infty \quad \text{for some } p > 1, \tag{1.2b}$$

$$\inf_{H>0} P(H) + \Theta_0 > 0. \tag{1.2c}$$

Then there exists a unique solution to (1.1) *such that $H_y > 0$ in $\mathbb{R}_+$. Furthermore*

$$H(y) = \alpha y^{\frac{2}{m+1}}(1 + o(1)), \quad H_y(y) = \tfrac{2\alpha}{m+1} y^{\frac{1-m}{m+1}}(1 + o(1)) \quad \text{as } y \to 0, \quad \alpha = \left(\tfrac{A(m+1)^2}{2(m-1)}\right)^{\frac{1}{m+1}}, \tag{1.3}$$

and $B > 0$ exists such that

$$\tfrac{1}{3U} H_y^3(y(H)) = L(H) + \ln B + O\left(\tfrac{\ln\ln H}{\ln H}\right) \quad \text{as } H \to +\infty, \quad L(H) := \ln H - \tfrac{1}{3}\ln\ln H. \tag{1.4}$$

Note from (1.3) that the linear-log fronts in Theorem 1.1 have $H_y(0) = +\infty$, i.e., a microscopic contact angle equal to $\frac{\pi}{2}$ at the contact line. For a discussion on the admissibility of this feature in the framework of lubrication approximation we refer to [14, Remark 1.2].

Remark 1.2. For the thin-film equation, two fundamental quantities are the energy E and its rate of dissipation D (see e.g. the discussion in [14]). Near the contact line, the advancing fronts identified in Theorem 1.1 have finite rate of energy dissipation for all $m > 1$, in the sense that

$$D[H] := \int_0^1 H^3 \left(H_{yy} - P'(H)\right)_y^2 \mathrm{d}y \overset{(1.1a)}{=} \int_0^1 \frac{U^2}{H(y)} \mathrm{d}y \overset{(1.3)}{<} +\infty,$$

and finite energy if $m < 3$, in the sense that

$$E[H] := \int_0^1 \left(\tfrac{1}{2}H_y^2 + P(H) - S\right) \mathrm{d}y < +\infty \quad \overset{(1.3)}{\Longleftrightarrow} \quad m < 3$$

(note that both H_y^2 and $P(H)$ scale like $y^{\frac{2(1-m)}{m+1}}$).

Remark 1.3. If $P > 0$ in $\mathbb{R}_+$ then Theorem 1.1 holds for all $\Theta_0 > 0$, hence in particular for all $S < 0$ (regardless of the values of $\hat{\mu}_{\mathrm{CL}} \geq 0$ and $U > 0$ in (1.1d)). Since $P' < 0$ (together with (1.2)) implies $P > 0$, Theorem 1.1 fully covers the partial wetting case with a purely repulsive potential. It also partially covers the case of long-range attractive potentials (i.e. $P'(H) > 0$ for $H \gg 1$), but then (1.2c) implies a non-generic upper bound on S (for instance, $S < \inf P < 0$ if $\hat{\mu}_{\mathrm{CL}} = 0$).

In view of the two remarks above, Theorem 1.1 confirms that mildly repulsive potentials ($1 < m < 3$) effectively resolve the no-slip paradox. It also confirms that (1.1d) acts as a selection criterion for fronts at the contact line, as much as the microscopic contact angle does in the case of slippage models [7].

Remark 1.4. As already observed in [20, 14, 24], (1.4) sets the length-scale in the Cox-Hocking-Voinov relation [43, 42, 9, 28, 29] between speed and macroscopic contact angle (see the discussion in [17]). Indeed, after simple computations it translates back into

$$H_y^3(y) = 3U \ln\left((3U)^{1/3} B y\right) + o(1) \quad \text{as } y \to +\infty. \tag{1.5}$$

Note that B may depend on Θ_0 (and in fact it appears to do it, see Figure 1 below).

For the proof of Theorem 1.1 we capitalize, as customary in this matter, on translation invariance of (1.1a) in order to reduce the order of the ODE: letting

$$\psi(H) = \tfrac{1}{2}H_y^2(y(H)), \qquad w(H) = \psi(H) - P(H), \tag{1.6}$$

(1.1a) and (1.1d) read respectively as

$$w''(H) = -\frac{U}{H^2\sqrt{2(P(H) + w(H))}} \quad \text{for } H \in \mathbb{R}_+, \qquad w(0) = \Theta_0. \tag{1.7a}$$

It will turn out to be sufficient to relax (1.1c) to $H_{yy}(+\infty) = 0$, which under (1.6) turns into $\psi'(+\infty) = 0$, which in view of (1.2b) means that

$$w'(+\infty) = 0. \tag{1.7b}$$

In Section 2 we use a fixed point argument to show that (1.7) admits a unique solution, and in Section 3 we translate this finding back to $H(y)$. The main point of this procedure is to show that

$$C^{-1} f(H) \leq \psi(H) \leq C f(H), \qquad f(H) = \begin{cases} P(H) + \Theta_0 & \text{for } H \ll 1 \\ \ln^{2/3}(e + H) & \text{for } H \gg 1 \end{cases} \tag{1.8}$$

for some $C > 1$ (see (2.8)-(2.9) below), consistently with the two regimes in (1.3) and (1.4). Assumption (1.2c) is crucial to obtain the lower bound. The bounds and the fixed point argument are based on the representation formula

$$\psi(H) = P(H) + \Theta_0 + \int_0^{+\infty} \frac{U \min\{\eta, H\}}{\eta^2 \sqrt{2\psi(\eta)}} \mathrm{d}\eta. \tag{1.9}$$

Once the solution is obtained, the bounds will a-posteriori be upgraded to the more precise asymptotics (1.3) and (1.4). In fact, (1.4) will follow from an application of earlier results in [20].

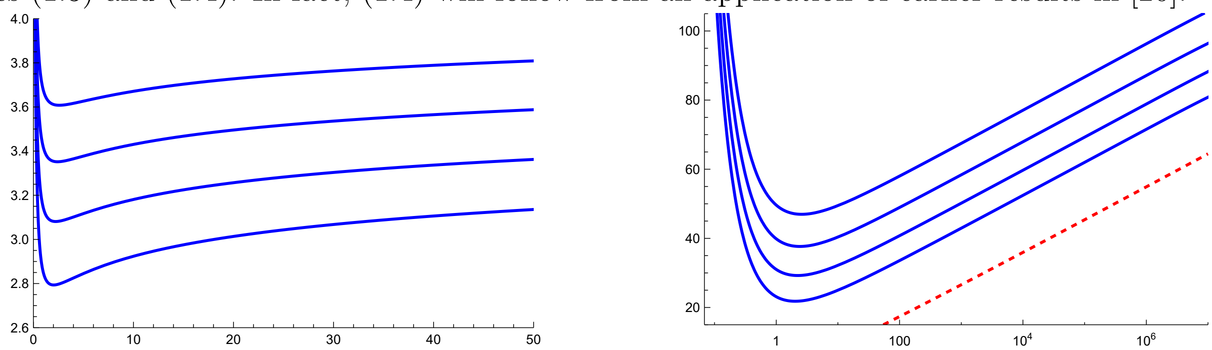

FIGURE 1. Approximate numerical solutions to (1.1) with $P(H) = H^{-1}$ ($m = 2$, $A = 1$) and $U = \sqrt{2}$ for $\Theta_0 = 1, 2, 3, 4$ (bottom to top). On the left, plots of H_y as a function of H; on the right, log-linear plots of H_y^3 as a function of H (solid) and the asymptotic function $3UL(H)$ (dashed, cf. (1.4)).

Figure 1 reports approximate numerical solutions to (1.1) with $P(H) = H^{-1}$ and $U = \sqrt{2}$ for different values of $\Theta_0 > 0$. They are obtained by solving the equation in (1.7a) in $(\varepsilon, \varepsilon^{-1})$ with $w(\varepsilon) = \Theta_0$, $|w'(\varepsilon^{-1})| < \varepsilon$, and $\varepsilon = 10^{-8}$. It is apparent that $P(H)$ generates a boundary layer near $H = 0$ and that Θ_0 macroscopically acts as a translation of $H_y^3(y(H))$, whence determining the constant B in (1.4) and (1.5).

Being limited to $\Theta_0 > 0$ by (1.2b) and (1.2c), Theorem 1.1 leaves the case $S \geq 0$ generically open (it only covers $S < \hat{\mu}_{\mathrm{CL}} U$ for purely repulsive potentials). On the other hand, the numerical evidences in [14], as well as of course the heuristics in [11], suggest that for negative values Θ_0 with $|\Theta_0|$ sufficiently large a *precursor region* appears before the front reaches its "macroscopic" linear-log profile. This points to the presence, if $\Theta_0 \leq 0$, of a third, intermediate regime which modifies the bounds in (1.8), and other tools besides the representation formula (1.9) might be necessary to cover this case.

In addition to the one we just discussed, the analysis of contact-line motion for the thin-film equation with null slippage and (mildly) repulsive potentials leaves quite a few other relevant questions open, for which we refer to [14, Section 4]. Here we just mention global well-posedness and stability for the full evolution problem with initial data close to the fronts in Theorem 1.1, in the spirit of [6, 19, 32].

2. WELL-POSEDNESS FOR (1.7)

By a solution to (1.7) we mean a function $w \in C^2(\mathbb{R}_+) \cap C^0(\overline{\mathbb{R}}_+)$ such that $P + w > 0$ in $\mathbb{R}_+$ and (1.7) hold. Theorem 1.1 will be a consequence of the following two results, which are in fact more general in terms of regularity and behaviour of P.

Theorem 2.1. *Let $U > 0$ and $P \in C(\mathbb{R}_+)$. For any $\Theta_0 \in \mathbb{R}$ there exists at most one solution w to* (1.7).

Theorem 2.2. *Let $U > 0$, $\Theta_0 \in \mathbb{R}$, and $P \in C(\mathbb{R}_+)$ such that* (1.2a) *and* (1.2c) *hold, together with*

$$\limsup_{H\to+\infty} |P(H)| < +\infty. \tag{2.1}$$

Then there exists a solution to (1.7)*. Furthermore*

$$\inf_{\mathbb{R}_+}(P+w) > 0, \qquad \limsup_{H\to+\infty} \frac{P(H)+w(H)}{\ln^{2/3} H} < +\infty, \tag{2.2}$$

and

$$w'(H) = \begin{cases} \frac{2U}{3-m}\left(\frac{m-1}{2A}\right)^{1/2} H^{\frac{m-3}{2}}(1+o(1)) & \text{if } m<3, \\ U\left(\frac{1}{A}\right)^{1/2}(\ln H)(1+o(1)) & \text{if } m=3, \\ O(1) & \text{if } m>3, \end{cases} \quad \text{as } H \to 0. \tag{2.3}$$

If in addition $P \in C^1(\mathbb{R}_+)$ and $P'(H) = O(H^{-p})$ as $H \to +\infty$ for some $p > 1$, then

$$(P+w)' > 0 \ \text{ for all } H \text{ sufficiently large.} \tag{2.4}$$

Proof of Theorem 2.1. Assume that w_1, w_2 are two solutions to (1.7) in $\mathbb{R}_+$ and let $w = w_1 - w_2$. Then w solves

$$w'' = -\frac{U}{\sqrt{2}H^2}\left(\frac{1}{\sqrt{P+w_1}} - \frac{1}{\sqrt{P+w_2}}\right) \quad \text{with } w(0) = w'(+\infty) = 0.$$

Since $w \mapsto \frac{1}{\sqrt{P+w}}$ is monotone decreasing, we have

$$ww'' \geq 0 \quad \text{in } \mathbb{R}_+. \tag{2.5}$$

Thus $(w^2)'' = 2(w')^2 + 2ww'' \geq 0$. Assume by contradiction that $w^2(H_0) > 0$ at some $H_0 \in \mathbb{R}_+$. Since $w^2(0) = 0$, there exists $H_1 \in (0, H_0)$ such that $(w^2)'(H_1) > 0$. Then $(w^2)'' \geq 0$ implies that $(w^2)'(H) \geq (w^2)'(H_1) > 0$ for all $H \geq H_1$, hence $w^2 > 0$ and $(w^2)' > 0$ for all $H \geq H_0$. Assume without losing generality that $w > 0$ in $[H_0, +\infty)$. Then (2.5) entails $w'' \geq 0$ in $[H_0, +\infty)$, and the inequality $0 < (w^2)' = 2ww'$ implies that also $w' > 0$ in $[H_0, +\infty)$. Therefore $w'(+\infty) > 0$, in contradiction with the assumption $w'(+\infty) = 0$. □

Proof of Theorem 2.2. We let $u = U/\sqrt{2}$ for notational convenience. We wish to apply a Schauder fixed point argument in C^0 spaces. Since we expect that $w'(0) = +\infty$ for $m < 3$ (cf. (2.3)) and that $w(+\infty) = +\infty$ (cf. (1.4)), we introduce approximating problems defined on $(\varepsilon, \varepsilon^{-1})$, $\varepsilon \in (0,1)$:

$$\begin{cases} w''(H) = -\dfrac{u}{H^2\sqrt{P(H)+w(H)}} & \text{for } H \in I_\varepsilon := (\varepsilon, \varepsilon^{-1}), \\ w(\varepsilon) = \Theta_0, \ w'(\varepsilon^{-1}) = 0. \end{cases} \tag{2.6}$$

We notice that any $w \in C^2(\bar{I}_\varepsilon)$ with $w(\varepsilon) = \Theta_0$ and $w'(\varepsilon^{-1}) = 0$ satisfies

$$\begin{aligned} w(H) - \Theta_0 &= \int_\varepsilon^H \int_\eta^{1/\varepsilon} -w''(\eta')\mathrm{d}\eta'\mathrm{d}\eta = -\int_\varepsilon^H \mathrm{d}\eta' \int_\varepsilon^{\eta'} \mathrm{d}\eta\, w''(\eta') - \int_H^{1/\varepsilon} \mathrm{d}\eta' \int_\varepsilon^H \mathrm{d}\eta\, w''(\eta') \\ &= -\int_\varepsilon^{1/\varepsilon} (\eta' \wedge H - \varepsilon) w''(\eta')\mathrm{d}\eta', \end{aligned}$$

where we let $a \wedge b = \min\{a,b\}$. Hence we consider the solution operator

$$\tilde{w} \mapsto T_\varepsilon(\tilde{w}) = w = \Theta_0 + v, \quad v(H) := \int_\varepsilon^{1/\varepsilon} \frac{u(\eta \wedge H - \varepsilon)}{\eta^2\sqrt{P(\eta)+\tilde{w}(\eta)}}\mathrm{d}\eta \tag{2.7}$$

in the space

$$\mathcal{S}_\varepsilon := \left\{ w \in C(\bar{I}_\varepsilon) : \ M^{-1}f(H) \leq P + w \leq Mf(H) \right\}, \tag{2.8}$$

where $M>1$ will be chosen later and

$$f(H):=\begin{cases} P(H)+\Theta_0 & \text{if } H\le H_0,\\ \ln^{2/3}(e+H) & \text{if } H>H_0,\end{cases}\qquad H_0=\sup\Big\{H:\ P(\eta)+\Theta_0>\ln^{2/3}(e+\eta)\ \ \forall \eta<H\Big\} \tag{2.9}$$

(note that $H_0\in(0,+\infty)$ in view of (1.2a) and (2.1)). The choice of f is motivated by the expected leading order behaviours of $H(y)$, hence of $w(H)$ (cf. (1.3) and (1.4)). By definition $P(H)+\Theta_0>0$ in $(0,H_0]$, hence by (1.2a) and (2.1) there exists $c_0,c_1\ge 1$ such that

$$c_0^{-1}H^{1-m}\le P(H)+\Theta_0\le c_0H^{1-m}\quad \text{for } H\in(0,H_0),\qquad P(H)+\Theta_0\le c_1\quad \text{for } H>H_0. \tag{2.10}$$

In particular f has a positive minimum in $\mathbb{R}_+$, thus both $\sqrt{P+\tilde w}$ and the integral on the right-hand side of (2.7) are well defined for all $\tilde w\in\mathcal{S}_\varepsilon$. It is clear that $\mathcal{S}_\varepsilon$ is a closed, bounded and convex subspace of $C^0(\overline{I}_\varepsilon)$ for all $\varepsilon\in(0,1)$.

We will consider any $\varepsilon>0$ such that $\varepsilon<\frac{H_0}{4}\wedge\frac{1}{H_0}$, and we will choose $M>1$ sufficiently large. Throughout the rest of the proof, $C\ge 1$ denotes a generic constant which is independent of M and ε, and we write $a\lesssim b$, resp. $a\ll b$, whenever one such C exists so that $a\le Cb$, resp. $Ca\le b$. Let $\tilde w\in\mathcal{S}_\varepsilon$ and v,w as defined in (2.7).

Step 1: $T_\varepsilon(\mathcal{S}_\varepsilon)\subset\mathcal{S}_\varepsilon$. We will first prove the upper bound:

$$\exists M_1\gg 1:\ P+w\le Mf\quad \text{in } (\varepsilon,\tfrac{1}{\varepsilon})\ \text{ for all } M\ge M_1. \tag{2.11}$$

We write

$$\frac{v(H)}{u\sqrt{M}}\overset{(2.8)}{\le}\int_0^{H_0}\frac{\eta\wedge H}{\eta^2\sqrt{P(\eta)+\Theta_0}}\mathrm{d}\eta+\int_{H_0}^{+\infty}\frac{\eta\wedge H}{\eta^2\ln^{1/3}(e+\eta)}\mathrm{d}\eta.$$

For $H\in[\varepsilon,H_0]$ we estimate

$$\begin{aligned}\frac{v(H)}{u\sqrt{M}}&\le\int_0^{H}\frac{1}{\eta\sqrt{P(\eta)+\Theta_0}}\mathrm{d}\eta+\int_H^{H_0}\frac{H}{\eta^2\sqrt{P(\eta)+\Theta_0}}\mathrm{d}\eta+\int_{H_0}^{+\infty}\frac{H}{\eta^2\ln^{1/3}(e+\eta)}\mathrm{d}\eta\\ &\overset{(2.10)}{\lesssim}\int_0^{H}\eta^{\frac{m-3}{2}}\mathrm{d}\eta+\int_H^{H_0}H\eta^{\frac{m-5}{2}}\mathrm{d}\eta+\int_{H_0}^{+\infty}\frac{H}{\eta^2\ln^{1/3}(e+\eta)}\mathrm{d}\eta.\end{aligned}$$

Since $H\le H_0$, we have

$$\int_H^{H_0}H\eta^{\frac{m-5}{2}}\mathrm{d}\eta\lesssim\begin{cases} H\left|H^{\frac{m-3}{2}}-H_0^{\frac{m-3}{2}}\right|\le H_0^{\frac{m-1}{2}} & \text{if } m>1,\ m\ne 3\\ H\ln\frac{H_0}{H}\le\frac{H_0}{e} & \text{if } m=3\end{cases} \tag{2.12}$$

and

$$\int_0^H\eta^{\frac{m-3}{2}}\mathrm{d}\eta\lesssim H^{\frac{m-1}{2}}\le H_0^{\frac{m-1}{2}},\qquad \int_{H_0}^{+\infty}\frac{H}{\eta^2\ln^{1/3}(e+\eta)}\mathrm{d}\eta\le\int_{H_0}^{+\infty}\frac{H}{\eta^2}\mathrm{d}\eta\le\frac{H}{H_0}\le 1.$$

Therefore $\frac{v(H)}{u\sqrt{M}}\lesssim 1$ for $H\le H_0$, which implies that

$$P+w=P+\Theta_0+v\le P+\Theta_0+C\sqrt{M}\quad \text{in } (\varepsilon,H_0).$$

Hence in (ε,H_0) the inequality $P+w\le Mf=M(P+\Theta_0)$ is implied by $(P+\Theta_0)(M-1)\ge C\sqrt{M}$, which holds true for M sufficiently large in view of (2.10). This proves the inequality in (2.11) for $H\in(\varepsilon,H_0)$. For $H\ge H_0$ we analogously estimate

$$\frac{v(H)}{u\sqrt{M}}\le\int_0^{H_0}\frac{1}{\eta\sqrt{P(\eta)+\Theta_0}}\mathrm{d}\eta+\int_{H_0}^{H}\frac{1}{\eta\ln^{1/3}(e+\eta)}\mathrm{d}\eta+\int_H^{+\infty}\frac{H}{\eta^2\ln^{1/3}(e+\eta)}\mathrm{d}\eta.$$

For the middle integral, we note that $\frac{1}{\eta} = \frac{1}{e+\eta} + \frac{e}{\eta(e+\eta)} < \frac{1}{e+\eta} + \frac{e}{\eta^2}$, so that

$$\int_{H_0}^{H} \frac{1}{\eta \ln^{1/3}(e+\eta)} \mathrm{d}\eta \le \tfrac{3}{2} \ln^{2/3}(e+H) + \frac{e}{H_0 \ln^{1/3}(e+H_0)}.$$

Using also $(2.10)_1$, we see that

$$\frac{v(H)}{u\sqrt{M}} \lesssim H_0^{\frac{m-1}{2}} + \ln^{2/3}(e+H) + \frac{1}{H_0 \ln^{1/3}(e+H_0)} + \frac{1}{\ln^{1/3}(e+H_0)} \lesssim 1 + \ln^{2/3}(e+H).$$

This means that

$$P + w \overset{(2.10)_2}{\le} c_1 + C\sqrt{M}(1 + \ln^{2/3}(e+H)) \lesssim \sqrt{M}(1 + \ln^{2/3}(e+H)) \quad \text{in } (H_0, \tfrac{1}{\varepsilon}).$$

Therefore for $H \in (H_0, \frac{1}{\varepsilon})$ we have

$$\begin{aligned} P + w \le Mf(H) = M\ln^{2/3}(e+H) \;&\Longleftarrow\; C\sqrt{M}(1 + \ln^{2/3}(e+H)) \le M \ln^{2/3}(e+H) \\ &\Longleftrightarrow\; C \le (\sqrt{M} - C)\ln^{2/3}(e+H), \end{aligned}$$

and the latter holds true for all $H \ge H_0$ choosing M sufficiently large. This completes the proof of (2.11).

We now prove the lower bound, where the crucial assumption (1.2c) is used:

$$\exists M_2 \gg 1 : \;\; P + w \ge \tfrac{1}{M} f \;\; \text{in } (\varepsilon, \tfrac{1}{\varepsilon}) \quad \text{for all } M \ge M_2. \tag{2.13}$$

For $H < H_0$, it suffices to note that by definition $v \ge 0$: $P + w = P + \Theta_0 + v \ge P + \Theta_0 = f$, hence the lower bound in fact holds for all $M \ge 1$. For $H > H_0$ we write

$$\frac{\sqrt{M} v(H)}{u} \ge \int_{\varepsilon}^{H_0} \frac{\eta - \varepsilon}{\eta^2 \sqrt{P(\eta) + \Theta_0}} \mathrm{d}\eta + \int_{H_0}^{H} \frac{\eta - \varepsilon}{\eta^2 \ln^{1/3}(e+\eta)} \mathrm{d}\eta + \int_{H}^{1/\varepsilon} \frac{H - \varepsilon}{\eta^2 \ln^{1/3}(e+\eta)} \mathrm{d}\eta.$$

We note that $\frac{\eta-\varepsilon}{\eta^2} > \frac{1}{2\eta} > \frac{1}{2(e+\eta)}$ if $\eta > 2\varepsilon$ (recall that $4\varepsilon < H_0$). Using these inequalities in the first two integrals (and neglecting the third one, which is non-negative) we obtain

$$\begin{aligned} \frac{\sqrt{M} v(H)}{u} &\overset{(2.10)}{\gtrsim} \int_{2\varepsilon}^{H_0} \eta^{\frac{m-3}{2}} \mathrm{d}\eta + \int_{H_0}^{H} \frac{1}{(e+\eta)\ln^{1/3}(e+\eta)} \mathrm{d}\eta \\ &\gtrsim H_0^{\frac{m-1}{2}} - (2\varepsilon)^{\frac{m-1}{2}} + \ln^{2/3}(e+H) - \ln^{2/3}(e+H_0) \\ &\gtrsim H_0^{\frac{m-1}{2}} + \ln^{2/3}(e+H) - \ln^{2/3}(e+H_0), \end{aligned}$$

where in the last inequality we used that $4\varepsilon < H_0$. Therefore the inequality in (2.13) holds true in $(H_0, \frac{1}{\varepsilon})$ if

$$P(H) + \Theta_0 + \frac{1}{C\sqrt{M}} \left(H_0^{\frac{m-1}{2}} + \ln^{2/3}(e+H) - \ln^{2/3}(e+H_0) \right) - \frac{1}{M} \ln^{2/3}(e+H) \ge 0$$

for all $H \ge H_0$. We rewrite this inequality as

$$\left[P(H) + \Theta_0 + \frac{1}{C\sqrt{M}} \left(H_0^{\frac{m-1}{2}} - \ln^{2/3}(e+H_0) \right) \right] + \left(\frac{1}{C\sqrt{M}} - \frac{1}{M} \right) \ln^{2/3}(e+H) \ge 0,$$

from which we see that because of (1.2c) both summands are positive for M sufficiently large. Thus (2.13) holds.

Step 2: existence of w_ε. Now that M has been chosen once for all (independently of ε), there is no further need to track it explicitly; hence C will denote a generic constant greater than 1 independent of ε, and the notation $a \lesssim b$ will be used accordingly. We now show that $T_\varepsilon(\mathcal{S}_\varepsilon)$ is relatively compact in $\mathcal{S}_\varepsilon$. Note that by definition

$$0 \le w'(H) = v'(H) = \int_H^{1/\varepsilon} \frac{u}{\eta^2\sqrt{P(\eta)+\tilde{w}(\eta)}}\mathrm{d}\eta \overset{(2.8)}{\lesssim} \int_H^{1/\varepsilon} \frac{u}{\eta^2\sqrt{f(\eta)}}\mathrm{d}\eta.$$

Therefore, using (2.9),

$$0 \le w'(H) \lesssim \int_H^{1/\varepsilon} \frac{1}{\eta^2 \ln^{1/3}(e+\eta)}\mathrm{d}\eta \le \frac{1}{H} \le \frac{1}{H_0} \lesssim 1 \quad \text{for } H \in (H_0, \tfrac{1}{\varepsilon}), \tag{2.14a}$$

$$0 \le w'(H) \overset{(2.10)}{\lesssim} \int_H^{H_0} \eta^{\frac{m-5}{2}}\mathrm{d}\eta + \frac{1}{H_0} \overset{(2.12)}{\lesssim} \max\{\tfrac{1}{H_0}, \ln\tfrac{H_0}{H}, |H^{\frac{m-3}{2}} - H_0^{\frac{m-3}{2}}|\} \quad \text{for } H \in (\varepsilon, H_0), \tag{2.14b}$$

which imply relative compactness in $\mathcal{S}_\varepsilon$ by Arzelà–Ascoli theorem. By the Schauder's Theorem, there exists a fixed point $w_\varepsilon \in \mathcal{S}_\varepsilon$:

$$w_\varepsilon(H) = \Theta_0 + \int_0^{+\infty} \chi_{I_\varepsilon}(\eta) \frac{u(\eta \wedge H - \varepsilon)}{\eta^2\sqrt{P(\eta)+w_\varepsilon(\eta)}}\mathrm{d}\eta \qquad \text{for all } H \in I_\varepsilon, \tag{2.15}$$

where χ_I denotes the characteristic function of I.

Step 3: the limit $\varepsilon \to 0$. It follows from (2.8) and (2.14) that $\|w_\varepsilon\|_{C^1(I_\varepsilon)} \le C_\varepsilon$. Therefore, by a standard diagonal procedure and Arzelà–Ascoli theorem, a subsequence (not relabeled) exists such that $w_\varepsilon \to w \in C(\mathbb{R}_+)$ locally uniformly, and (2.8) implies that

$$f(H) \lesssim P(H) + w(H) \lesssim f(H) \quad \text{for all } H \in \mathbb{R}_+ \tag{2.16}$$

(recall that M has been chosen once for all, independently of ε). We now pass to the limit $\varepsilon \to 0$ in (2.15). For any fixed $H \in \mathbb{R}_+$ we have

$$\chi_{I_\varepsilon}(\eta)\frac{(\eta \wedge H - \varepsilon)}{\eta^2\sqrt{P(\eta)+w(\eta)}} \overset{(2.16)}{\lesssim} \frac{\eta \wedge H}{\eta^2\sqrt{f(\eta)}} \overset{(2.9)}{\lesssim} \left\{ \begin{array}{ll} \eta^{\frac{m-3}{2}} & \text{if } \eta \le \min\{H_0, H\} \\ \frac{H}{\eta^2 \ln^{1/3}(e+\eta)} & \text{if } \eta \ge \max\{H_0, H\} \\ O(1) & \text{otherwise} \end{array} \right\} \in L^1(\mathbb{R}_+).$$

Hence by dominated convergence we conclude that

$$w(H) = \Theta_0 + \int_0^{+\infty} \frac{u(\eta \wedge H)}{\eta^2\sqrt{P(\eta)+w(\eta)}}\mathrm{d}\eta \qquad \text{for all } H \in \mathbb{R}_+. \tag{2.17}$$

Equation (2.17) implies that $w \in C^2(\mathbb{R}_+)$ and that w satisfies (1.7a), and condition (1.7b) follows from (2.14a). The bounds in (2.16) imply in particular that (2.2) holds. The asymptotic in (2.3) follows from integration of (1.7a) around $H = 0$. It remains to prove (2.4). We have

$$w'(H) \overset{(2.17)}{=} \int_H^{+\infty} \frac{u}{\eta^2\sqrt{P(\eta)+w(\eta)}}\mathrm{d}\eta \overset{(2.16)}{\gtrsim} \int_H^{+\infty} \frac{\mathrm{d}\eta}{\eta^2 \ln^{1/3}(e+\eta)} \quad \text{for } H > H_0.$$

Since $\ln(e+\eta) = \ln\eta + O(\eta^{-1})$ as $\eta \to +\infty$, we have in particular $\ln(e+\eta) \lesssim \ln\eta$ for $\eta \gg 1$. Then

$$w'(H) \gtrsim J := \int_H^{+\infty} \frac{\mathrm{d}\eta}{\eta^2 \ln^{1/3}\eta} \quad \text{for } H \gg 1.$$

On the other hand, an integration by parts shows that

$$J = \frac{1}{H\ln^{1/3} H} - \frac{1}{3}\int_H^{+\infty} \frac{\mathrm{d}\eta}{\eta^2 \ln^{4/3}\eta} \ge \frac{1}{H\ln^{1/3} H} - \frac{J}{3\ln H} \quad \text{for } H \gg 1.$$

Hence $J \gtrsim (H\ln^{1/3} H)^{-1}$ for $H \gg 1$, which implies that $w'(H) \gtrsim (H\ln^{1/3} H)^{-1}$ for $H \gg 1$. Since by assumption $P'(H) = O(H^{-p})$ as $H \to +\infty$ for some $p > 1$, this bound implies that $(P+w)' > 0$ for H sufficiently large, i.e. (2.4). □

3. Proof of Theorem 1.1

Here we show how the main result of this note follows from Theorem 2.1 and Theorem 2.2.

Proof of Theorem 1.1.

Existence. Assumption (2.1) is obviously satisfied in view of (1.2b). Therefore, by Theorem 2.2 there exists a solution w to (1.7). Noting that the change of variables (1.6) entails $H_y = \pm\sqrt{2(P+w)}$, we undo it by defining

$$H(y) = F^{-1}(y), \quad F(H) = \int_0^H \frac{1}{\sqrt{2(P(\eta)+w(\eta))}}\mathrm{d}\eta, \tag{3.1}$$

so that $H_y > 0$ in $\mathbb{R}_+$ and $H(0) = 0$ (recall that $P + w$ is bounded from below in $\mathbb{R}_+$ in view of $(2.2)_1$), thus proving (1.1b). Since $w(0) = \Theta_0$, we have

$$F(H) = \sqrt{\tfrac{2(m-1)}{A(m+1)^2}}\,H^{\frac{m+1}{2}}(1+o(1)) \quad \text{as } H \to 0.$$

Hence F is well defined and

$$H(y) = \left(\tfrac{A(m+1)^2}{2(m-1)}\right)^{\frac{1}{m+1}} y^{\frac{2}{m+1}}(1+o(1)) \quad \text{as } y \to 0.$$

Using again $H_y(y) = \sqrt{2P(H(y))}(1 + o(1))$ as $y \to 0$ we obtain (1.3). Since $w \in C^2(\mathbb{R}_+)$ solves the equation in (1.7a), $H \in C^3(\mathbb{R}_+)$ solves (1.1a), and (1.1d) follows from $w(0) = \Theta_0$.

It remains to show (1.4), which entails (1.1c). For that, we notice that $\psi = P + w$ satisfies

$$H^2\psi''(H) = H^2P''(H) - \frac{U}{\sqrt{2\psi(H)}} = \frac{U}{\sqrt{2\psi(H)}}(-1+f(H)), \quad f(H) = \tfrac{1}{U}H^2P''(H)\sqrt{2\psi(H)}.$$

Because of $(2.2)_2$ and (1.2b), $H^2P''(H)\sqrt{\psi(H)} = O(H^{1-p}\ln^{1/3} H)$ as $H \to +\infty$. Hence we may apply the arguments in [20, Section 4]: letting

$$u = \frac{(2\psi)^{3/2}}{3U}, \qquad s = \ln H,$$

the previous equation turns into

$$\frac{3}{2}u^{1/3}\left(\frac{\mathrm{d}}{\mathrm{d}s} - 1\right)\frac{\mathrm{d}u^{2/3}}{\mathrm{d}s} - 1 + f(s), \quad f(s) = O(s^{1/3}e^{s(1-p)}) \quad \text{as } s \to +\infty. \tag{3.2}$$

One can easily check that the arguments in the proof of [20, Proposition 4.1] can be repeated line by line, provided that $H_y > 0$, $\psi' > 0$ for H sufficiently large, and $f(s) \lesssim s^{-2}\ln s$ as $s \to +\infty$ (see also the paragraph above Proposition 4.1 in [20]). Now, $H_y > 0$ in $\mathbb{R}_+$ follows from (3.1), $\psi' > 0$ for H sufficiently large holds true in view of (2.4) (which holds true thanks to (1.2b)), and the asymptotic property of f is immediate from (3.2). Hence [20, Proposition 4.1] applies and yields existence of $a \in \mathbb{R}$ such that $u(s) = s - \frac{1}{3}\ln s + a + O(s^{-1}\ln s)$ as $s \to +\infty$. This translates into

$$\tfrac{1}{3U}(2\psi(H))^{3/2} = \ln H - \tfrac{1}{3}\ln\ln H + a + O(\tfrac{\ln\ln H}{\ln H}) \quad \text{as } H \to +\infty,$$

which coincides with (1.4) setting $B = e^a > 0$.

Uniqueness. Let H be any solution to (1.1) with $H_y > 0$ in $\mathbb{R}_+$. Then the inverse $y(H)$ is well defined. In view of (1.1c),

$$\frac{H(y)}{y(3U\ln y)^{1/3}} \to 1 \ \text{ as } y \to +\infty, \quad \text{i.e.} \quad \frac{y(H)}{H(3U\ln H)^{-1/3}} \to 1 \ \text{ as } H \to +\infty. \tag{3.3}$$

Let $\psi(H) = \frac{1}{2}(H_y(y(H))^2$. Since $P \in C^2(\mathbb{R}_+)$, the function $w(H) = \frac{1}{2}(H_y(y(H))^2 - P(H)$ is of class $C^2(\mathbb{R}_+)$ and satisfies (1.7a). In view of $(3.3)_1$, we have

$$H^2P''(H)H_y \overset{(1.2b)}{=} O(H^{1-p})H_y \overset{(1.1c)}{=} O((y\ln^{1/3} y)^{1-p})O(\ln^{1/3} y) = o(1) \quad \text{as } y \to +\infty,$$

hence $H^2H_{yyy} = -U + H^2P''(H)H_y = -U(1 + o(1))$ as $y \to +\infty$. Therefore, using again (3.3),

$$H_{yyy} = -Uy^{-2}(3U\ln y)^{-2/3}(1 + o(1)) \quad \text{and} \quad H_{yy} = Uy^{-1}(3U\ln y)^{-2/3}(1 + o(1)) \quad \text{as } y \to +\infty.$$

Recalling (1.2b) and using $(3.3)_2$, this means that $w'(H) = H_{yy}(y(H)) - P'(H) \to 0$ as $H \to +\infty$, which implies (1.7b). Thus w is a solution to (1.7). Since w is unique in view of Theorem 2.1, the conclusion follows using $H(0) = 0$: since $H_y(y) = \sqrt{2(P(H(y)) + w(H(y)))}$, H is uniquely determined by (3.1). □

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