AN INTEGRAL FORMULA FOR THE INHOMOGENEOUS JORDAN–VON NEUMANN EQUATION

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ABSTRACT. We study the inhomogeneous form of the Jordan–von Neumann quadratic functional equation, in which the right-hand side is a prescribed function g of two real variables. We prove that the existence of a C^2 solution is equivalent to g being itself of class C^2 and satisfying a single three-variable cocycle identity, and we exhibit the solution as a closed-form integral expression involving the second partial derivative of g along the first coordinate axis. The construction preserves regularity along the standard scale of C^k, smooth, and polynomial classes.

1. INTRODUCTION

The Jordan–von Neumann functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \qquad x,y\in\mathbb{R}, \tag{1}$$

is the algebraic skeleton of the parallelogram identity and, by the celebrated theorem of Jordan and von Neumann [6], characterizes those norms on a real vector space that arise from an inner product. Its solution space admits an explicit description: by Kurepa [8, 9], every $f\colon \mathbb{R}\to\mathbb{R}$ satisfying (1) is of the form $f(x)=B(x,x)$ for a unique $\mathbb{Q}$-bilinear symmetric form $B\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and under any of several mild regularity hypotheses (e.g., continuity at one point, Lebesgue measurability, or boundedness on a set of positive measure), this collapses to $f(x)=cx^2$ for some $c\in\mathbb{R}$. Comprehensive accounts of this theory are given in the monographs of Aczél and Dhombres [1] and Kuczma [7].

In the present paper we are concerned with the *inhomogeneous* version of (1): for a prescribed function $g\colon \mathbb{R}^2\to\mathbb{R}$, we ask when there exists $f\colon \mathbb{R}\to\mathbb{R}$ such that

$$f(x+y)+f(x-y)-2f(x)-2f(y)=g(x,y), \qquad (x,y)\in\mathbb{R}^2, \tag{2}$$

and how f is then determined by g. We denote by Q the linear operator $f\mapsto \mathsf{Q}[f]$ defined by the left-hand side of (2), which now reads $\mathsf{Q}[f]=g$; the kernel $\ker\mathsf{Q}$ is exactly the solution space of (1).

The question is natural from several perspectives. From the perspective of the stability theory of functional equations initiated by Hyers and Ulam—and developed for the quadratic equation by Skof [11], Cholewa [2], and Czerwik [3]; see also [5]—the data g represents a controlled perturbation of (1), and the recovery of f from g is the exact inverse of the perturbation map. From an algebraic standpoint, the question

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parallels the classical inverse problem for the linear Cauchy operator $\mathsf{L}[f](x,y) := f(x+y) - f(x) - f(y)$. For L, it is a standard fact that g belongs to the image of L if and only if g satisfies both the cocycle identity

$$g(x,y) + g(x+y,z) = g(x,y+z) + g(y,z), \qquad x,y,z \in \mathbb{R}, \tag{3}$$

and the symmetry condition $g(x,y) = g(y,x)$; identity (3) is the standard 2-cocycle equation that arises in group cohomology (see, e.g., [1]) and is forced by the associativity of addition. Erdős [4] showed that the situation is sensitive to regularity: without any regularity hypothesis, the symmetry condition does not follow from the cocycle identity, as he established by exhibiting an asymmetric solution of (3) via a Hamel-basis construction; under continuity of g, however, the cocycle identity (3) implies the symmetry of g, and every continuous g satisfying (3) is then in the image of L. For $g \in C^1(\mathbb{R}^2)$, Prunescu [10] gave a constructive counterpart that exhibits an explicit solution.

We refer to (3) as the cocycle identity for L. The analog for the quadratic operator Q is the three-variable identity

$$g(x+y,z) + g(x-y,z) + 2g(x,y) = g(x,y+z) + g(x,y-z) + 2g(y,z), \tag{4}$$

holding for all $x,y,z \in \mathbb{R}$ whenever $g \in \operatorname{Im}\mathsf{Q}$. The derivation is parallel to that of (3) for L but relies on the parallelogram structure of Q rather than on the associativity of addition; see Section 2. By analogy with the linear case, we call identity (4) the cocycle identity for Q.

Our main result, stated below, asserts that, in the class C^2, identity (4) is both necessary and sufficient for g to lie in the image of Q, and that a solution is then given by a closed-form integral.

Theorem 1.1. *Let $g\colon \mathbb{R}^2 \to \mathbb{R}$.*

(i) *If $g = \mathsf{Q}[f]$ for some $f\colon \mathbb{R} \to \mathbb{R}$, then g satisfies the cocycle identity (4).*

(ii) *Conversely, if $g \in C^2(\mathbb{R}^2)$ satisfies (4), then the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = -\tfrac{1}{2}\, g(0,0) - \tfrac{1}{2}\,(\partial_2 g)(0,0)\,x + \tfrac{1}{2}\int_0^x (x-t)\,(\partial_2^2 g)(t,0)\,\mathrm{d}t \tag{5}$$

belongs to $C^2(\mathbb{R})$ and satisfies $\mathsf{Q}[f] = g$ on $\mathbb{R}^2$.

The solution is unique up to addition of an element of $\ker\mathsf{Q}$.

Throughout, ∂_1 and ∂_2 denote partial differentiation in the first and second variables, respectively. The argument for part (ii) proceeds by differentiating the cocycle identity (4) in z at $z = 0$. One differentiation yields that $(\partial_2 g)(\cdot,0)$ is constant on $\mathbb{R}$; two differentiations produce an identity satisfied by $h(t) := (\partial_2^2 g)(t,0)$ that algebraically reproduces $\mathsf{Q}[f]$ for any f with $f'' = h/2$. Two integrations then give the formula (5). The construction is the analog for Q of the integral formula obtained by Prunescu [10] for L in the class C^1.

2. THE COCYCLE IDENTITY AND ITS CONSEQUENCES

2.1. Necessity.

Proof of Theorem 1.1 (i). Set $g = \mathsf{Q}[f]$. We show that both sides of (4) expand to

$$S(x,y,z) := f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) \\ - 4f(x) - 4f(y) - 4f(z). \quad (6)$$

For the left-hand side, sum

$$\begin{aligned} g(x+y,z) &= f(x+y+z) + f(x+y-z) - 2f(x+y) - 2f(z), \\ g(x-y,z) &= f(x-y+z) + f(x-y-z) - 2f(x-y) - 2f(z), \\ 2g(x,y) &= 2f(x+y) + 2f(x-y) - 4f(x) - 4f(y); \end{aligned}$$

the $\pm 2f(x+y)$ and $\pm 2f(x-y)$ terms cancel, and the result is $S(x,y,z)$. For the right-hand side, the analogous expansion of $g(x,y+z) + g(x,y-z) + 2g(y,z)$ produces the same expression $S(x,y,z)$. □

2.2. Operatorial reading of the cocycle identity.

The cocycle identities (3) and (4) admit a unifying operatorial description that places them in a general framework. Let $T\colon \mathscr{F}(\mathbb{R},\mathbb{R}) \to \mathscr{F}(\mathbb{R}^2,\mathbb{R})$ be a linear operator measuring the defect of some property compatible with the group structure of $(\mathbb{R},+)$, in the sense that $T[f] \equiv 0$ characterizes that property (here $\mathscr{F}(\mathbb{R}^n,\mathbb{R})$ denotes the space of real-valued functions on $\mathbb{R}^n$). The iterated application of T along the two coordinate axes of $T[f]$ produces two three-variable defects: at z fixed, the slice $t \mapsto T[f](t,z)$ has its own defect $T\big[T[f](\cdot,z)\big](x,y)$; at x fixed, the slice $t \mapsto T[f](x,t)$ has the defect $T\big[T[f](x,\cdot)\big](y,z)$. The *iterated cocycle identity*

$$T\big[\,T[f](\,\cdot\,,z)\,\big](x,y) \;=\; T\big[\,T[f](x,\,\cdot\,)\,\big](y,z), \qquad x,y,z \in \mathbb{R}, \quad (7)$$

asserts that these two iterated defects coincide: the iterated defect of $T[f]$ is axis-symmetric. This is the precise content of the cocycle condition for any such T, and it specializes to the standard 2-cocycle identity for $T = \mathsf{L}$ as well as to (4) for $T = \mathsf{Q}$.

2.3. Consequences of the cocycle identity.

Lemma 2.1. *Suppose $g\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies the cocycle identity (4). Then:*

(i) $g(x,0) = g(0,0)$ *for every* $x \in \mathbb{R}$*;*
(ii) *if* $g \in C^1(\mathbb{R}^2)$*, then* $(\partial_2 g)(x,0) = (\partial_2 g)(0,0)$ *for every* $x \in \mathbb{R}$*;*
(iii) *if* $g \in C^2(\mathbb{R}^2)$*, then, with* $h(t) = (\partial_2^2 g)(t,0)$*,*

$$h(x+y) + h(x-y) - 2h(y) = 2(\partial_2^2 g)(x,y), \qquad (x,y) \in \mathbb{R}^2. \quad (8)$$

Proof. (i) Set $y = 0$ and $z = 0$ in (4) and obtain $4g(x,0) = 2g(x,0) + 2g(0,0)$, so $g(x,0) = g(0,0)$.

(ii) Differentiate (4) in z at $z = 0$. By the chain rule, the terms $g(x,y\pm z)$ on the right-hand side contribute $\pm(\partial_2 g)(x,y)$ and cancel, while $g(x\pm y,z)$ and $g(y,z)$ contribute, respectively, $(\partial_2 g)(x\pm y,0)$ and $(\partial_2 g)(y,0)$. The resulting identity reads

$$(\partial_2 g)(x+y,0) + (\partial_2 g)(x-y,0) = 2(\partial_2 g)(y,0). \quad (9)$$

Setting $y = 0$ in (9) gives the claim.

(iii) Differentiate (4) twice in z at $z=0$. By the chain rule, each term $g(x,y\pm z)$ contributes $(\partial_2^2 g)(x,y)$ on the right-hand side; the remaining terms produce $h(x\pm y)$ on the left and $2h(y)$ on the right. $\square$

3. The integral formula

Proof of Theorem 1.1 (ii). Let $h(t)=(\partial_2^2 g)(t,0)$ and f be defined by (5), i.e.,

$$f(x)=-\tfrac{1}{2}\,g(0,0)-\tfrac{1}{2}\,(\partial_2 g)(0,0)\,x+\tfrac{1}{2}\int_0^x (x-t)\,h(t)\,\mathrm{d}t.$$

Since $h\in C(\mathbb{R})$, the function f is well-defined and belongs to $C^2(\mathbb{R})$, with

$$f(0)=-\tfrac{1}{2}\,g(0,0),\qquad f'(0)=-\tfrac{1}{2}\,(\partial_2 g)(0,0),\qquad f''(x)=\tfrac{1}{2}\,h(x). \tag{10}$$

Set $E(x,y)=\mathsf{Q}[f](x,y)-g(x,y)$. By linearity of ∂_2^2 and the definition of Q,

$$\begin{aligned}(\partial_2^2\mathsf{Q}[f])(x,y)&=f''(x+y)+f''(x-y)-2f''(y)\\&=\tfrac{1}{2}\big[h(x+y)+h(x-y)-2h(y)\big],\end{aligned}$$

which equals $(\partial_2^2 g)(x,y)$ by Lemma 2.1 (iii). Hence, $\partial_2^2 E\equiv 0$, so

$$E(x,y)=A(x)+yB(x)$$

for some $A,B\colon\mathbb{R}\to\mathbb{R}$. By Lemma 2.1 (i), $g(x,0)=g(0,0)$, and by (10), $\mathsf{Q}[f](x,0)=-2f(0)=g(0,0)$. Hence, $A(x)=E(x,0)=0$. Differentiating $E(x,y)=yB(x)$ in y gives $B(x)=(\partial_2 E)(x,0)=(\partial_2\mathsf{Q}[f])(x,0)-(\partial_2 g)(x,0)$. We compute

$$(\partial_2\mathsf{Q}[f])(x,0)=f'(x)-f'(x)-2f'(0)=-2f'(0)=(\partial_2 g)(0,0)$$

by (10), and $(\partial_2 g)(x,0)=(\partial_2 g)(0,0)$ by Lemma 2.1 (ii). Hence, $B\equiv 0$, $E\equiv 0$, and $\mathsf{Q}[f]=g$.

Uniqueness modulo $\ker\mathsf{Q}$ is the standard consequence of the linearity of Q. $\square$

Remark 3.1. The representation (5) can also be written as an iterated double integral:

$$f(x)=-\tfrac{1}{2}\,g(0,0)-\tfrac{1}{2}\,(\partial_2 g)(0,0)\,x+\tfrac{1}{2}\int_0^x\!\int_0^v(\partial_2^2 g)(t,0)\,\mathrm{d}t\,\mathrm{d}v; \tag{11}$$

the equivalence with (5) follows by integration by parts or by Fubini's theorem.

Corollary 3.2. *Suppose g satisfies the cocycle identity (4) and belongs to one of the regularity classes $C^k(\mathbb{R}^2)$ (with $k\geq 2$), $C^\infty(\mathbb{R}^2)$, or the class of real polynomials in two variables. Then the function f defined by (5) belongs to the corresponding class $C^k(\mathbb{R})$, $C^\infty(\mathbb{R})$, or the class of real polynomials in one variable, respectively.*

Proof. The function $h=(\partial_2^2 g)(\cdot,0)$ lies in the corresponding one-variable regularity class, hence so does $f''=h/2$, and hence so does f by the iterated-integral form (11). If g is a real polynomial in two variables of total degree d, then h is a real polynomial in one variable of degree at most d, and f is a real polynomial in one variable of degree at most $d+2$. $\square$

4. Examples and further directions

We illustrate and verify Theorem 1.1 with some examples of functions g that satisfy the cocycle identity (4) and the corresponding solutions.

Example 4.1. Take $g(x,y) = x^2y^2$, $(x,y) \in \mathbb{R}^2$, which satisfies (4) (both sides equal $2(x^2y^2 + x^2z^2 + y^2z^2)$), hence Theorem 1.1 applies. With $g(0,0) = 0$, $(\partial_2 g)(0,0) = 0$, $h(t) = (\partial_2^2 g)(t,0) = 2t^2$, (5) yields

$$f(x) = \tfrac{1}{2}\int_0^x (x-t)\cdot 2t^2\,\mathrm{d}t = \tfrac{x^4}{3} - \tfrac{x^4}{4} = \tfrac{x^4}{12},$$

so $g = \mathsf{Q}\left[\frac{x^4}{12}\right]$.

Example 4.2. Take $g(x,y) = 2\,\mathrm{e}^x\cosh y - 2\,\mathrm{e}^x - 2\,\mathrm{e}^y$, $(x,y) \in \mathbb{R}^2$, which satisfies (4) since $g = \mathsf{Q}[\mathrm{e}^x]$. Then $g(0,0) = -2$, $(\partial_2 g)(0,0) = -2$, $h(t) = 2\,\mathrm{e}^t - 2$, and (5) yields

$$\begin{aligned} f(x) &= 1 + x + \tfrac{1}{2}\int_0^x (x-t)\,(2\,\mathrm{e}^t - 2)\,\mathrm{d}t \\ &= 1 + x + (\mathrm{e}^x - x - 1) - \tfrac{x^2}{2} = \mathrm{e}^x - \tfrac{x^2}{2}, \end{aligned}$$

which is e^x modulo the kernel element $-\frac{1}{2}x^2 \in \ker \mathsf{Q}$.

Example 4.3. Take $g(x,y) = 2\cos x\cos y - 2\cos x - 2\cos y$, $(x,y) \in \mathbb{R}^2$, which satisfies (4) since $g = \mathsf{Q}[\cos x]$. Then $g(0,0) = -2$, $(\partial_2 g)(0,0) = 0$, $h(t) = 2 - 2\cos t$, and (5) yields

$$f(x) = 1 + \tfrac{1}{2}\int_0^x (x-t)\,(2 - 2\cos t)\,\mathrm{d}t = 1 + \tfrac{x^2}{2} + (\cos x - 1) = \cos x + \tfrac{x^2}{2},$$

which is $\cos x$ modulo the kernel element $\frac{1}{2}x^2 \in \ker \mathsf{Q}$.

Further directions. *The purely algebraic problem.* Whether the cocycle identity (4) alone, with no regularity hypothesis on g, is sufficient to ensure $g \in \operatorname{Im}\mathsf{Q}$, is open. The corresponding question for the linear operator L has a definitive answer: Erdős [4] showed that the linear cocycle identity (3) must be supplemented by a symmetry condition $g(x,y) = g(y,x)$, without which Hamel-basis constructions yield solutions of (3) that are not in the image of L. A parallel obstruction may exist for Q and deserves separate study.

The continuous case. For the linear operator L, Erdős [4] proved that every continuous solution of the cocycle identity (3) is automatically symmetric, hence lies in the image of L. Whether the cocycle identity (4) for Q admits an analogous conclusion under mere continuity of g—without the C^2 hypothesis used here—is an attractive open question.

The C^1 case. The formula (5) requires $g \in C^2(\mathbb{R}^2)$ in order to define $h = (\partial_2^2 g)(\cdot,0)$. For $g \in C^1(\mathbb{R}^2)$ satisfying (4), only one differentiation of the cocycle is available, and a closed-form solution would have to rely on a different combination of g, $\partial_1 g$ and $\partial_2 g$. This would parallel the C^1 formula obtained by Prunescu [10] for L and seems worth pursuing.

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