

A bilinear approach to the finite field restriction problem, II

Mark Lewko

ABSTRACT. Let P_3 denote the three-dimensional paraboloid over a finite field of prime order in which -1 is not a square. We prove that the Fourier extension operator associated with P_3 maps L^2 to L^r for $r > \frac{176}{51} = 3.45098\dots$. The argument combines the author's bilinear approach to the problem with point-line incidence estimates. We also prove that the extension operator associated with the paraboloid P_6 in six dimensions maps L^2 to $L^{8/3}$. This was previously known up to but not including the endpoint, and is the sharp L^2 estimate in six dimensions. Finally we observe that the endpoint restriction conjecture for P_3 in finite fields implies that the integer lattice points on the 3-d Euclidean paraboloid are a $\Lambda(3)$ set.

1. Introduction

The Fourier restriction problem over finite fields was introduced by Mockenhaupt and Tao [15] and has since been studied extensively, with connections to additive combinatorics, incidence geometry, Kakeya phenomena, the theory of quadratic forms, and explicit extractor constructions, among other topics. We refer the reader to [11] and [15] for an introduction to the problem, and to [6, 7, 10, 12, 17, 19] for some of the more recent developments.

Let F be a finite field of odd characteristic, with $q = |F|$, and let

$$P_d = \{(\underline{x}, \underline{x} \cdot \underline{x}) : \underline{x} \in F^{d-1}\} \subset F^d$$

denote the d -dimensional paraboloid, endowed with the normalized counting measure $d\sigma$ that assigns mass $|P_d|^{-1} = q^{-(d-1)}$ to each point. For $f : P_d \rightarrow \mathbb{C}$ the Fourier extension operator is given by

$$(fd\sigma)^\vee(x) = q^{-(d-1)} \sum_{\xi \in P_d} f(\xi) e(x \cdot \xi), \quad x \in F^d,$$

where e is a fixed nontrivial additive character of F . This work will focus entirely on the L^2 theory. We write $R^*(2 \rightarrow r)$ for the best constant in

$$\|(fd\sigma)^\vee\|_{L^r(F^d)} \leq R^*(2 \rightarrow r) \|f\|_{L^2(P_d, d\sigma)},$$

with the implied constant required to be independent of q . By duality this is equivalent to the inequality $\|\widehat{g}\|_{L^2(P_d, d\sigma)} \lesssim \|g\|_{L^{r'}(F^d)}$, with $r' = \frac{r}{r-1}$.

We begin with the three-dimensional paraboloid. The Stein–Tomas method gives $R^*(2 \rightarrow r) \lesssim 1$ for $r \geq 4$, which is the optimal L^2 range over an arbitrary finite field. When -1 is not a square in F the surface contains no lines, and one expects a larger range; in this setting Mockenhaupt and Tao [15] proved $R^*(2 \rightarrow r) \lesssim 1$ for $r > \frac{18}{5} = 3.6$ and conjectured that the inequality holds for every $r \geq 3$. Their range was improved to $r > \frac{18}{5} - \delta$, for some small $\delta > 0$, in arbitrary fields of odd characteristic, and to $r \geq \frac{745}{207} \approx 3.599$ in prime fields, by the author [12]. In prime fields the range was subsequently lowered to $r > \frac{68}{19} \approx 3.579$ by Stevens and de Zeeuw [19], to $r \geq \frac{32}{9} \approx 3.555$ by

Rudnev and Shkredov [17], and to $r > \frac{188}{53} \approx 3.547$ by the author [10]. In arbitrary fields of odd characteristic in which -1 is not a square, the first paper in this series [9] gives $r > \frac{32}{9} = 3.555\dots$ ¹

These improvements have followed two distinct paths. Beginning from the Mockenhaupt–Tao exponent $r > \frac{18}{5}$, nearly all of the progress has come from sharpening the additive-energy input in the Mockenhaupt–Tao argument. That input is bounded using point-line incidence estimates, and ultimately Rudnev’s point-plane incidence theorem [16], which in turn rests on a result of Kollár [8] in algebraic geometry. Incidence bounds of this type are sensitive to the presence of subfields, so the resulting restriction estimates have been confined to prime fields; see the survey [18]. The second and more recent idea is the bilinear approach of [9], which replaces the Mockenhaupt–Tao argument by an estimate for a related bilinear operator, of which the Mockenhaupt–Tao argument is essentially the diagonal. This approach has the advantage of working in arbitrary fields of odd characteristic. Here we combine the two approaches, to prove the following:

THEOREM 1.1. *Let $F = \mathbb{F}_p$ be a field of prime order in which -1 is not a square, and let $P_3 = \{(x, x \cdot x) : x \in F^2\}$. Then $R^*(2 \rightarrow r) \lesssim_r 1$ for every*

$$r > \frac{176}{51} = 3.45098\dots$$

Our second result concerns the paraboloid in six dimensions. In [7], Iosevich, Koh and the author established the optimal L^2 restriction estimate for P_d in every even dimension $d \geq 8$, and in dimension six proved $R^*(2 \rightarrow \frac{8}{3} + \varepsilon) \lesssim 1$ for every $\varepsilon > 0$, which is sharp apart from the endpoint. Here we obtain the endpoint.

THEOREM 1.2. *Let F be a finite field of odd characteristic and let $P_6 = \{(\xi, \xi \cdot \xi) : \xi \in F^5\} \subset F^6$. Then*

$$R^*\left(2 \rightarrow \frac{8}{3}\right) \lesssim 1, \quad \text{equivalently} \quad \|(fd\sigma)^\vee\|_{L^{8/3}(F^6)} \lesssim \|f\|_{L^2(P_6, d\sigma)}.$$

In the final section we record a transference observation. The conjectured endpoint $L^2 \rightarrow L^3$ restriction estimates for the paraboloid and the sphere in F^3 would imply that the integer paraboloid and the lattice spheres in \mathbb{Z}^3 are $\Lambda(3)$ sets, with a constant independent of the size of the support. This implication, with no power loss in the size of the support on the right-hand side, is specific to ambient dimension three and is perhaps some indication of the difficulty of the full finite field restriction problem. No such fully explicit $\Lambda(3)$ set that is not also a $\Lambda(4)$ set is presently known.

2. Background and notation

We use counting measure on F^d and normalized counting measure on the paraboloid, so for a set S we write 1_S for its indicator and $|S|$ for its cardinality. For $g : F^d \rightarrow \mathbb{C}$ we set $\widehat{g}(\xi) = \sum_{x \in F^d} g(x)e(-x \cdot \xi)$. We write $X \lesssim Y$ to mean $X \leq CY$ for a constant C independent of q , and $X \lesssim_\varepsilon Y$ when the constant is allowed to depend on a parameter ε ; the relation $X \sim Y$ means $X \lesssim Y$ and $Y \lesssim X$. For a function g the notation $g(x) \sim \lambda$ means $\lambda/2 \leq |g(x)| \leq 2\lambda$. We collect the various estimates we will need from the papers [7, 9, 11, 10, 15]; we state them without proof and refer to those papers for the details.

An estimate of the form $\|(fd\sigma)^\vee\|_{L^r(F^d)} \lesssim \|f\|_{L^2(P_d, d\sigma)}$ is dual to

$$(2.1) \quad \|\widehat{g}\|_{L^2(P_d, d\sigma)} \lesssim \|g\|_{L^{r'}(F^d)}, \quad r' = \frac{r}{r-1},$$

¹The published version of [9] stated the larger range $r > \frac{24}{7}$, owing to an error in the final dominance calculation, where a term of the form $q^{-3/16}|G|^{13/16}$ was taken to be dominated by $|G|^{17/24}$ throughout the relevant range, whereas this holds only for $|G| \leq q^{9/5}$; the worst case is at $|G| \sim q^2$ and gives $r > \frac{32}{9}$. This is discussed in an erratum accompanying the revised version of [9] on the arXiv, whose estimates we use here. That paper also remarked that one of the terms could be lowered using incidence estimates but, believing that term not to be dominant, stated this would not improve the main result; it is in fact the dominant term, and improving the term and the restriction estimate is the purpose of Section 3.

and one has the identity $\|\widehat{g}\|_{L^2(P_d, d\sigma)}^2 = \langle g, g * (d\sigma)^\vee \rangle_{F^d}$, the inner product being taken with respect to counting measure. Here $(d\sigma)^\vee = \delta_0 + K$, where for $x = (\underline{x}, x_d) \in F^{d-1} \times F$ with $x \neq 0$ the kernel is the Gauss sum

$$K(x) = q^{-(d-1)} \sum_{\underline{\xi} \in F^{d-1}} e(\underline{x} \cdot \underline{\xi} + x_d \underline{\xi} \cdot \underline{\xi}).$$

Writing $G(a) = \sum_{t \in F} e(at^2)$ for the one-dimensional Gauss sum, with $|G(a)| = q^{1/2}$ for $a \neq 0$, completing the square evaluates this as

$$K(x) = q^{-(d-1)} G(x_d)^{d-1} e\left(-\frac{\underline{x} \cdot \underline{x}}{4x_d}\right) \quad (x_d \neq 0), \quad K(x) = 0 \quad (x_d = 0, x \neq 0)$$

(see (18) of [15]). In particular $|K(x)| \leq q^{-(d-1)/2}$ for $x \neq 0$, which equals q^{-1} in the three-dimensional case $d = 3$. The decomposition yields

$$(2.2) \quad \|\widehat{g}\|_{L^2(P_d, d\sigma)} \leq \|g\|_{L^2(F^d)} + q^{-(d-1)/4} \|g\|_{L^1(F^d)},$$

which in dimension $d = 3$ gives the bound with coefficient $q^{-1/2}$. Interpolating with the Parseval bound $\|\widehat{g}\|_{L^2(P_d, d\sigma)} \leq q^{1/2} \|g\|_{L^2(F^d)}$ recovers the Stein–Tomas estimate.

The kernel K also relates the fourth moment of a convolution to the restriction of that convolution to horizontal slices. For $h : F^d \rightarrow \mathbb{C}$ and $z \in F$ write $h_z(\eta) = h(\eta, z)$ for the slice of h at height z , a function on F^{d-1} , and let \widetilde{h}_z be its lift to the paraboloid, the function on P_d defined by $\widetilde{h}_z(\eta, \eta \cdot \eta) = h_z(\eta)$.

LEMMA 2.1 ([7, Lemma 2.1]). *For every $h : F^d \rightarrow \mathbb{C}$,*

$$(2.3) \quad \|h * K\|_{L^4(F^d)} \lesssim q^{(d-1)/2} \sum_{z \in F} \|(\widetilde{h}_z d\sigma)^\vee\|_{L^4(F^d)}.$$

This reduces the fourth moment of $h * K$ to the extension operators of the planar slices \widetilde{h}_z , each of which is estimated by the additive energy of its support through Plancherel.

We use two standard reductions, both recalled in [9, 11]. The first is dyadic pigeonholing, which reduces matters, at the cost of a factor q^ε , to functions g with $g \sim 1$ on their support G . The second is the following epsilon-removal lemma.

LEMMA 2.2 ([11, Lemma 16]). *Let $s_0 > 1$, and suppose that for every $\varepsilon > 0$,*

$$(2.4) \quad \|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim_\varepsilon q^\varepsilon |G|^{1/s_0}$$

holds for every g with $g \sim 1$ on its support G . Then

$$(2.5) \quad \|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim_s \|g\|_{L^s(F^d)} \quad \text{for every } s < s_0.$$

By (2.1), the bound (2.5) is the extension estimate $R^*(2 \rightarrow r) \lesssim_r 1$ for every $r > s'_0$, where $s'_0 = \frac{s_0}{s_0-1}$. It therefore suffices to prove the restricted estimate (2.4) for $g \sim 1_G$.

For the combinatorial input we record three quantities. For a finite subset E of an abelian group, its additive energy is

$$(2.6) \quad \Lambda(E) = |\{(a, b, c, d) \in E^4 : a + b = c + d\}|.$$

For $A \subset F^2$ we call a triple (x_0, x_1, x_2) a corner if $(x_1 - x_0) \cdot (x_2 - x_1) = 0$, and a quadruple (x_0, x_1, x_2, x_3) a rectangle if each cyclically consecutive triple is a corner; $\mathcal{R}(A)$ denotes the number of rectangles in A . For $A, B \subset F^2$ we call (x_1, x_2, y_1, y_2) , with $x_1, x_2 \in A$ and $y_1, y_2 \in B$, a trapezoid if $x_1 \neq x_2$, $y_1 \neq y_2$, and $x_1 - x_2 = \lambda(y_1 - y_2)$ for some $\lambda \in F^\times$, that is, if the two differences are nonzero and parallel; $\mathcal{T}(A, B)$ denotes the number of trapezoids. Configurations with $x_1 = x_2$ or $y_1 = y_2$ are not counted. The following follows from Vinh’s incidence estimate [20] and can be found, for example, as (3.6) and (3.7) in [9]:

LEMMA 2.3. For every $A \subset F^2$,

$$(2.7) \quad \mathcal{R}(A) \lesssim |A|^{5/2},$$

and

$$(2.8) \quad \mathcal{R}(A) \lesssim q^{-1}|A|^3 + q^{1/2}|A|^2,$$

the second bound being the smaller for $q \leq |A| \leq q^2$.

The bound (2.7) is an elementary Cauchy-Schwarz argument implicit in [15], and (2.8) is an incidence estimate of Vinh [20] (see (15) of [17]).

LEMMA 2.4 ([10, Theorem 4]). If $F = \mathbb{F}_p$ and $|A| \leq p^{26/21}$, then for every $\varepsilon > 0$

$$(2.9) \quad \mathcal{R}(A) \lesssim_\varepsilon |A|^{99/41+\varepsilon}.$$

The conjectured optimal Szemerédi–Trotter theorem in prime fields would give (2.9) with exponent $2 + \varepsilon$.

It is convenient to organize a planar set according to its concentration on lines. Following [9], a set $A \subset F^2$ is k -regular, for $k \leq |A|^{1/2}$, if it is a disjoint union of $\sim k$ subsets, each lying on one of $\sim k$ distinct lines and containing $\sim |A|k^{-1}$ points, such that every line outside this frame of $\sim k$ lines meets A in at most k points. We call A irregular if every line meets A in at most $|A|^{1/2}$ points. The following records the estimates of Lemma 5.2 and Lemma 5.3 from [9].

LEMMA 2.5 ([9]). Every $A \subset F^2$ is a disjoint union of $O(\log q)$ sets, one k -regular for each dyadic $k \leq |A|^{1/2}$ together with one irregular set. If A and B are k -regular with $|A| \sim |B| \sim m$, or both irregular with $k \sim m^{1/2}$, then

$$(2.10) \quad \mathcal{R}(A) \lesssim m^2 k,$$

and

$$(2.11) \quad \mathcal{T}(A, B) \lesssim m^3 k + m^4 k^{-2}.$$

We next turn to the main bilinear tool from [9], which should be thought of as a more delicate form of Lemma 2.1. Let $P = P_3$, let $g \sim 1$ on $G \subset F^3$, and for $z \in F$ let g_z be the restriction of g to the slice at height z , with support $G_z \subset F^2$.

PROPOSITION 2.6 ([9, (4.2)]). With the notation above,

$$(2.12) \quad \|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim |G|^{1/2} + |G|^{3/8} \left(\sum_z \|g_z * K\|_4^2 + \sum_{z \neq z'} \|g_z * K g_{z'} * K\|_2 \right)^{1/4},$$

the linear term satisfies

$$(2.13) \quad \|g_z * K\|_4^4 \lesssim q^{-1} \mathcal{R}(G_z),$$

and the bilinear term satisfies

$$(2.14) \quad \|g_z * K g_{z'} * K\|_2^2 \lesssim q^{-1} |G_z| |G_{z'}| + q^{-2} \mathcal{T}(G_z, G_{z'}) \quad (z \neq z').$$

Set

$$(2.15) \quad \mathcal{B}(A, B) = \min \{ \mathcal{T}(A, B), q(\mathcal{R}(A)\mathcal{R}(B))^{1/2} \}.$$

Combining (2.14) with the Cauchy-Schwarz consequence of (2.13) gives $\|g_z * K g_{z'} * K\|_2^2 \lesssim q^{-1} |G_z| |G_{z'}| + q^{-2} \mathcal{B}(G_z, G_{z'})$. Inserting the linear and bilinear bounds into (2.12), for regular $g \sim 1_G$ with w nonempty slices each of size $m \sim |G|w^{-1}$, and bounding the diagonal slice contribution by $q|G|$, gives

$$(2.16) \quad \|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim |G|^{1/2} + |G|^{3/8} q^{-1/8} \left(\sum_z \mathcal{R}(G_z)^{1/2} + q^{-1/2} \sum_{z \neq z'} \mathcal{B}(G_z, G_{z'})^{1/2} + q|G| \right)^{1/4}.$$

Since \mathcal{T} counts only nondegenerate parallel configurations, the degenerate ones, numbering $\lesssim |G_z||G_{z'}|$, are already accounted for by the term $q^{-1}|G_z||G_{z'}|$ in (2.14), while the diagonal slices $z = z'$ are accounted for by the term $q|G|$ in (2.16).

The quantity (2.15) is effective because a pair of sets cannot have both many trapezoids and many rectangles. This is made quantitative by the following estimate of [9], which is the source of the arbitrary-field exponent $\frac{32}{9}$.

PROPOSITION 2.7 ([9, Prop. 5.4]). *Let $A, B \subset F^2$ be k -regular sets, or both irregular sets, with $|A| \sim |B| \sim m$. Then*

$$(2.17) \quad \mathcal{B}(A, B) \lesssim m^{8/3}q^{2/3} + m^{7/2}.$$

Inserting Proposition 2.7 into (2.16) and combining with the standard estimates yields, for regular $g \sim 1_G$,

$$(2.18) \quad \|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim |G|^{1/2} + |G|^{11/16}q^{-1/8} + q^{-3/16}|G|^{13/16} + |G|^{17/24} + q^{1/8}|G|^{5/8},$$

and from this the table

$$(2.19) \quad \|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim |G|^{1/2} + \begin{cases} |G|q^{-1/2}, & |G| \leq q^{12/7}, \\ |G|^{17/24}, & q^{12/7} \leq |G| \leq q^{9/5}, \\ q^{-3/16}|G|^{13/16}, & q^{9/5} \leq |G| \leq q^2, \\ q^{3/16}|G|^{5/8}, & q^2 \leq |G| \leq q^{5/2}, \\ q^{1/2}|G|^{1/2}, & q^{5/2} \leq |G| \leq q^3. \end{cases}$$

The table is to be read after the dyadic slice-size and regularity reductions; the triangle-inequality sum over the resulting $O(\log^2 q)$ pieces is absorbed into the q^ε loss used in Lemma 2.2. Every term of (2.19) is bounded by $|G|^{23/32}$, so $R^*(2 \rightarrow r) \lesssim_r 1$ for $r > \frac{32}{9}$ in arbitrary fields of odd characteristic in which -1 is not a square.

In prime fields the rectangle bound (2.9) improves the input still further, and the argument of [10] gives

$$(2.20) \quad \|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim_\varepsilon |G|^{1/2} + \begin{cases} |G|p^{-1/2}, & |G| \leq p^{94/53}, \\ p^{3/41}|G|^{111/164+\varepsilon}, & p^{94/53} \leq |G| \leq p^{75/34}, \\ p^{3/16}|G|^{5/8}, & p^{75/34} \leq |G| \leq p^{5/2}, \\ p^{1/2}|G|^{1/2}, & p^{5/2} \leq |G| \leq p^3. \end{cases}$$

This gives $R^*(2 \rightarrow r) \lesssim_r 1$ for every $r > \frac{188}{53}$, the prime-field estimate of [10].

3. The prime-field improvement

In this section $F = \mathbb{F}_p$ and -1 is not a square. The general field argument of Section 2 identifies the obstruction in (2.18) as the term $q^{-3/16}|G|^{13/16}$, which arises from the $m^{7/2}$ summand of Proposition 2.7. We improve this in prime fields by using a direction-sensitive point-line incidence theorem of Lund, Pham and Vinh [14].

Two nonzero vectors of F^2 have the same direction if one is a scalar multiple of the other, and a line has the direction of any nonzero vector along it. There are $p+1$ directions in all. For $A \subset F^2$ and a direction θ we set

$$(3.1) \quad E_\theta(A) = |\{(a, a') \in A^2 : a - a' \text{ has direction } \theta\}|,$$

which counts ordered pairs of distinct points, since 0 has no direction. With the trapezoid count of Section 2 restricted to nondegenerate configurations, this gives the identity $\mathcal{T}(A, B) =$

$\sum_{\theta} E_{\theta}(A)E_{\theta}(B)$. More generally, for a family \mathcal{L} of lines we write $E_{\theta,\mathcal{L}}(A)$ for the number of ordered pairs $(a, a') \in A^2$ that lie on a common line of \mathcal{L} of direction θ . We refer to $\sum_{\theta} E_{\theta,\mathcal{L}}(A)^2$, and to $\sum_{\theta} E_{\theta}(A)^2$ when \mathcal{L} is the family of all lines, as the directional energy of A . By Cauchy-Schwarz in θ the trapezoid count satisfies $\mathcal{T}(A, B) \leq (\sum_{\theta} E_{\theta}(A)^2)^{1/2} (\sum_{\theta} E_{\theta}(B)^2)^{1/2}$, so a bound on the directional energy can be used to bound the trapezoid count.

For a finite set $A \subset \mathbb{F}_p^2$ and a finite family \mathcal{L} of lines, write

$$I(A, \mathcal{L}) = |\{(a, \ell) \in A \times \mathcal{L} : a \in \ell\}|$$

for the number of incidences, and let t denote the number of distinct directions spanned by the lines of \mathcal{L} . We use two point-line incidence bounds, one of Lund, Pham and Vinh and one of Vinh.

LEMMA 3.1 ([14]). *Let $A \subset \mathbb{F}_p^2$ with $|A| = m$ and let \mathcal{L} be a family of L lines spanning t directions, with $t \lesssim p^2/m$. Then*

$$(3.2) \quad I(A, \mathcal{L}) \lesssim t^{2/9} L^{5/9} m^{7/9} + t^{1/2} L^{1/2} m^{1/2} + L^{2/3} m^{2/3} + L + m.$$

The direction-sensitive incidence theorem of Lund, Pham and Vinh [14, Theorem 1.5] gives the same estimate with

$$\min\{t^{3/10} L^{3/5} m^{7/10}, t^{2/9} L^{5/9} m^{7/9}\}$$

in place of the first term on the right side of (3.2). Since we only need an upper bound, we replace this minimum by its second entry. Next we recall Vinh's well-known incidence estimate.

LEMMA 3.2 ([20]). *For every $A \subset \mathbb{F}_p^2$ with $|A| = m$ and every family \mathcal{L} of L lines,*

$$(3.3) \quad I(A, \mathcal{L}) \lesssim \frac{mL}{p} + (pmL)^{1/2}.$$

We will apply (3.3) to line families spanning $t \gtrsim p^2/m$ directions, outside the range of (3.2). We now put this together to obtain an estimate on the directional energy.

LEMMA 3.3. *Let $A \subset \mathbb{F}_p^2$ with $|A| = m$, and let \mathcal{L} be a family of lines each meeting A in at most $m^{1/2}$ points. Then for every $\varepsilon > 0$,*

$$(3.4) \quad \sum_{\theta} E_{\theta,\mathcal{L}}(A)^2 \lesssim_{\varepsilon} p^{\varepsilon} \left(m^3 \min(m, p)^{1/3} + \frac{m^4}{p} + pm^2 \right).$$

PROOF. We may take \mathcal{L} to be the family of all lines meeting A in at most $m^{1/2}$ points, since enlarging \mathcal{L} only increases the left-hand side. Put $\mu = \min(m, p)$. We sort these lines into classes, bound the directional energy contributed by each class, and sum. For dyadic parameters j and s , the (j, s) -class consists of the lines meeting A in between j and $2j$ points whose direction contains between s and $2s$ such lines. There are $O(\log^2 m)$ classes, so up to a factor m^{ε} , which we absorb into p^{ε} since $m \leq p^2$, it suffices to bound the contribution of each class.

Fix a (j, s) -class, and let t be the number of directions it spans. It has $\sim s$ lines in each of these directions and $\sim ts$ lines in all, each containing $\sim j$ points of A . A single direction of the class contributes $\sim sj^2$ ordered pairs, so the class contributes

$$(3.5) \quad t (sj^2)^2 = ts^2 j^4$$

to $\sum_{\theta} E_{\theta,\mathcal{L}}(A)^2$. We will use three constraints repeatedly. There are at most $p + 1$ directions, so $t \leq p$; the s lines of a single direction are parallel and disjoint, so $sj \lesssim m$; and $j \leq m^{1/2}$ which follows from the hypothesis.

We first dispose of the lines meeting A in at most $\tau = \mu^{1/3}$ points. Writing \mathcal{L}^- for these lines and \mathcal{L}^+ for the rest, $E_{\theta,\mathcal{L}}(A) = E_{\theta,\mathcal{L}^-}(A) + E_{\theta,\mathcal{L}^+}(A)$ for every θ , so it suffices to bound the two parts separately. In a given direction the lines of \mathcal{L}^- are parallel and disjoint and together containing at

most m points, and each contributes at most τ ordered pairs per point, so $E_{\theta, \mathcal{L}^-}(A) \leq \tau m$. Since also $\sum_{\theta} E_{\theta, \mathcal{L}^-}(A) \leq m^2$,

$$(3.6) \quad \sum_{\theta} E_{\theta, \mathcal{L}^-}(A)^2 \leq \tau m \sum_{\theta} E_{\theta, \mathcal{L}^-}(A) \leq \tau m \cdot m^2 = m^3 \mu^{1/3}.$$

There remain the (j, s) -classes with $j \geq \tau$. In such a class each of the $\sim ts$ lines carries at least j points, so

$$(3.7) \quad I(A, \mathcal{L}) \gtrsim jts.$$

Whichever incidence bound applies, at least one term on its right side is $\gtrsim jts$. Suppose first that the class spans few directions, $t \lesssim p^2/m$, so that (3.2) applies. If the term $t^{2/9}(ts)^{5/9}m^{7/9}$ in (3.2) is $\gtrsim jts$, then

$$(3.8) \quad jts \lesssim t^{2/9}(ts)^{5/9}m^{7/9} \implies t \lesssim m^{7/2}j^{-9/2}s^{-2} \implies ts^2j^4 \lesssim m^{7/2}j^{-1/2}.$$

At the same time, Vinh's estimate (3.3) applies to this same class. If the term $mL/p = mts/p$ in (3.3) is $\gtrsim jts$, then $j \lesssim m/p$, and using $t \leq p$ and $sj \lesssim m$ gives

$$ts^2j^4 \leq p(m/j)^2j^4 = pm^2j^2 \lesssim \frac{m^4}{p}.$$

If the term $(pmL)^{1/2} = (pmts)^{1/2}$ in (3.3) is $\gtrsim jts$, then $ts \lesssim pm/j^2$, and hence, using $sj \lesssim m$,

$$ts^2j^4 \lesssim (pmj^{-2})(mj^{-1})j^4 = pm^2j.$$

Consequently, in the case where the term $t^{2/9}(ts)^{5/9}m^{7/9}$ in (3.2) is $\gtrsim jts$, we have

$$(3.9) \quad ts^2j^4 \lesssim \frac{m^4}{p} + \min\{m^{7/2}j^{-1/2}, pm^2j\}.$$

If $m \leq p$, then $j \geq \mu^{1/3} = m^{1/3}$ and the minimum in (3.9) is at most $m^{10/3} = m^3\mu^{1/3}$. If $m \geq p$, put $j_0 = mp^{-2/3}$. Since $j_0 \geq p^{1/3} = \mu^{1/3}$, either $j \leq j_0$, in which case $pm^2j \leq m^3p^{1/3}$, or $j \geq j_0$, in which case $m^{7/2}j^{-1/2} \leq m^3p^{1/3}$. Thus this case contributes at most

$$\frac{m^4}{p} + m^3\mu^{1/3}.$$

We next handle the other terms in (3.2). If the term $t^{1/2}(ts)^{1/2}m^{1/2}$ is $\gtrsim jts$, then $s \lesssim m/j^2$, so $ts^2j^4 \leq tm^2 \leq pm^2$. If the term $(ts)^{2/3}m^{2/3}$ is $\gtrsim jts$, then $ts \lesssim m^2j^{-3}$, so $ts^2j^4 \lesssim m^2(sj) \leq m^3 \leq m^3\mu^{1/3}$. If the term m is $\gtrsim jts$, then $ts^2j^4 = (jts)(sj^3) \leq m^3 \leq m^3\mu^{1/3}$, using $sj \lesssim m$ and $j \leq m^{1/2}$. Finally, the L term in (3.2) is $L = ts$ for the present class. If this term is $\gtrsim jts$, then $j \lesssim 1$; using $sj \lesssim m$ and $t \leq p$ gives $ts^2j^4 \lesssim ts^2 \leq t(m/j)^2 \lesssim pm^2$. Thus each class with few directions satisfies

$$(3.10) \quad ts^2j^4 \lesssim m^3\mu^{1/3} + \frac{m^4}{p} + pm^2.$$

Suppose instead that the class spans many directions, $t \gtrsim p^2/m$, which is outside the range of (3.2). Here (3.3) applies with $L = ts$, and by (3.7) one of its two terms is $\gtrsim jts$. If the term $mL/p = mts/p$ is $\gtrsim jts$, then $j \lesssim m/p$, so using $t \leq p$ and $sj \lesssim m$,

$$(3.11) \quad ts^2j^4 \leq p(m/j)^2j^4 = pm^2j^2 \leq \frac{m^4}{p}.$$

If the term $(pmL)^{1/2} = (pmts)^{1/2}$ is $\gtrsim jts$, then $ts \lesssim pm/j^2$, hence $s \lesssim m^2/(pj^2)$ by $t \gtrsim p^2/m$, and again $ts^2j^4 \leq p(m^2/(pj^2))^2j^4 = m^4/p$.

Summing the bound (3.6) together with the bounds (3.10) and (3.11) over the $O(\log^2 m)$ classes, and absorbing the logarithmic factor into p^ϵ , gives (3.4). \square

We now replace Proposition 2.7 in prime fields.

PROPOSITION 3.4. *Let $A, B \subset \mathbb{F}_p^2$ be k -regular sets, or both irregular sets, with $|A| \sim |B| \sim m$. Then for every $\varepsilon > 0$,*

$$(3.12) \quad \mathcal{B}(A, B) \lesssim_\varepsilon p^\varepsilon \left(m^{8/3} p^{2/3} + m^3 \min(m, p)^{1/3} + \frac{m^4}{p} + pm^2 \right).$$

PROOF. Set $k_* = m^{2/3} p^{-1/3}$ and $\mu = \min(m, p)$.

If A and B are k -regular with $k \leq k_*$, then (2.10) gives $\mathcal{R}(A), \mathcal{R}(B) \lesssim m^2 k$, so the second term of (2.15) already gives the result,

$$\mathcal{B}(A, B) \leq p(\mathcal{R}(A)\mathcal{R}(B))^{1/2} \lesssim pm^2 k \leq pm^2 k_* = m^{8/3} p^{2/3}.$$

Suppose instead that A and B are k -regular with $k \geq k_*$, and use the first term of (2.15), $\mathcal{B}(A, B) \leq \mathcal{T}(A, B) = \sum_\theta E_\theta(A)E_\theta(B)$. Call a pair of distinct points a frame pair if the two points lie on a common frame line and a non-frame pair otherwise, and let $F_\theta(A)$ and $E_{\theta, \mathcal{L}_A}(A)$ count, respectively, the frame and non-frame pairs of A of direction θ , where \mathcal{L}_A is the family of non-frame lines of A . Then $E_\theta(A) = F_\theta(A) + E_{\theta, \mathcal{L}_A}(A)$, and likewise for B , so expanding the product splits the trapezoid count into three parts,

$$(3.13) \quad \mathcal{T}(A, B) = \sum_\theta F_\theta(A)F_\theta(B) + \sum_\theta E_{\theta, \mathcal{L}_A}(A)E_{\theta, \mathcal{L}_B}(B) + \sum_\theta (F_\theta(A)E_{\theta, \mathcal{L}_B}(B) + E_{\theta, \mathcal{L}_A}(A)F_\theta(B)),$$

the frame, non-frame, and mixed contributions, which we bound in turn. The non-frame lines \mathcal{L}_A and \mathcal{L}_B meet their sets in at most $k \leq m^{1/2}$ points. The frame of A consists of $\sim k$ lines with $\sim m/k$ points each, so the total number of frame pairs is $\sum_\theta F_\theta(A) \lesssim k(m/k)^2 = m^2/k$, and in particular

$$(3.14) \quad \left(\sum_\theta F_\theta(A)^2 \right)^{1/2} \leq \sum_\theta F_\theta(A) \lesssim \frac{m^2}{k},$$

and likewise for B .

For the frame contribution, Cauchy-Schwarz in θ and (3.14) give

$$\sum_\theta F_\theta(A)F_\theta(B) \leq \left(\sum_\theta F_\theta(A)^2 \right)^{1/2} \left(\sum_\theta F_\theta(B)^2 \right)^{1/2} \lesssim \frac{m^4}{k^2} \leq \frac{m^4}{k_*^2} = m^{8/3} p^{2/3}.$$

For the non-frame contribution, Lemma 3.3 applied to (A, \mathcal{L}_A) and to (B, \mathcal{L}_B) and Cauchy-Schwarz in θ give

$$\sum_\theta E_{\theta, \mathcal{L}_A}(A)E_{\theta, \mathcal{L}_B}(B) \leq \left(\sum_\theta E_{\theta, \mathcal{L}_A}(A)^2 \right)^{1/2} \left(\sum_\theta E_{\theta, \mathcal{L}_B}(B)^2 \right)^{1/2} \lesssim_\varepsilon p^\varepsilon \left(m^3 \mu^{1/3} + \frac{m^4}{p} + pm^2 \right).$$

For the mixed contribution, combining (3.14) with the bound of Lemma 3.3 for (B, \mathcal{L}_B) in the form

$$\left(\sum_\theta E_{\theta, \mathcal{L}_B}(B)^2 \right)^{1/2} \lesssim_\varepsilon p^\varepsilon \left(m^{3/2} \mu^{1/6} + m^2 p^{-1/2} + p^{1/2} m \right)$$

through Cauchy-Schwarz gives

$$(3.15) \quad \sum_\theta F_\theta(A)E_{\theta, \mathcal{L}_B}(B) \lesssim_\varepsilon p^\varepsilon \frac{m^2}{k} \left(m^{3/2} \mu^{1/6} + m^2 p^{-1/2} + p^{1/2} m \right).$$

Using $k \geq k_* = m^{2/3} p^{-1/3}$, the three summands in (3.15) are at most

$$(3.16) \quad m^{17/6} p^{1/3} \mu^{1/6}, \quad m^{10/3} p^{-1/6}, \quad m^{7/3} p^{5/6},$$

respectively, and each is dominated by the right side of (3.12). For the first term, if $m \leq p$ it is $m^3 p^{1/3} \leq m^{8/3} p^{2/3}$, while if $m \geq p$ it is at most $m^3 p^{1/3} = m^3 \mu^{1/3}$. For the second term, if $m \leq p$ it is at most $m^{10/3} = m^3 \mu^{1/3}$; if $p \leq m \leq p^{3/2}$ it is at most $m^3 p^{1/3} = m^3 \mu^{1/3}$; and if $m \geq p^{3/2}$ it is at most m^4/p . For the third term, $m^{7/3} p^{5/6} \leq m^{8/3} p^{2/3} + pm^2$ according as $m \geq p^{1/2}$ or $m \leq p^{1/2}$. The other mixed term, with the roles of A and B exchanged, is identical. Combining the three contributions gives (3.12) for regular sets.

Finally, suppose that A and B are irregular. Let \mathcal{L} be the family of all lines in \mathbb{F}_p^2 . Since every line meets A and B in at most $m^{1/2}$ points, Lemma 3.3 applies to both (A, \mathcal{L}) and (B, \mathcal{L}) . Also, because \mathcal{L} is the family of all lines, every nonzero pair of points of A or B is counted in exactly one line of \mathcal{L} . Thus

$$\mathcal{T}(A, B) = \sum_{\theta} E_{\theta}(A)E_{\theta}(B) = \sum_{\theta} E_{\theta, \mathcal{L}}(A)E_{\theta, \mathcal{L}}(B).$$

By Cauchy-Schwarz in θ and Lemma 3.3,

$$\begin{aligned} \mathcal{T}(A, B) &\leq \left(\sum_{\theta} E_{\theta, \mathcal{L}}(A)^2 \right)^{1/2} \left(\sum_{\theta} E_{\theta, \mathcal{L}}(B)^2 \right)^{1/2} \\ &\lesssim_{\varepsilon} p^{\varepsilon} \left(m^3 \mu^{1/3} + \frac{m^4}{p} + pm^2 \right), \end{aligned}$$

where we used $|A| \sim |B| \sim m$ in the last line. Since $\mathcal{B}(A, B) \leq \mathcal{T}(A, B)$, this proves the irregular case. \square

The preceding estimate controls the trapezoid side of (2.15). At the scale that will be critical below, one can also improve the rectangle side by retaining the dependence on p in the rich-line estimates, rather than passing directly to the estimate (2.9).

LEMMA 3.5 (Hybrid rectangle estimate). *Suppose that -1 is not a square in \mathbb{F}_p , and let $A \subset \mathbb{F}_p^2$ have cardinality n with*

$$p \leq n \leq p^{10/7}.$$

Then, for every $\varepsilon > 0$,

$$(3.17) \quad \mathcal{R}(A) \lesssim_{\varepsilon} p^{\varepsilon} n^2 p^{3/7}.$$

PROOF. We first dispose of degenerate rectangles. If one of the four edge differences is zero, say $x_0 = x_1$, then the two remaining corner conditions involving the opposite edge give $(x_3 - x_2) \cdot (x_3 - x_2) = 0$. Since -1 is not a square, the form $u \cdot u$ has no nonzero zeros, and hence $x_2 = x_3$. Thus degenerate rectangles are determined by two points of A , up to cyclic relabeling, and contribute $O(n^2)$, which is acceptable since $n \geq p$. We may therefore count only rectangles whose edge-lines are genuine lines.

For a dyadic integer k , let \mathcal{L}_k denote the lines containing between k and $2k$ points of A . The Stevens–de Zeeuw incidence theorem gives

$$(3.18) \quad |\mathcal{L}_k| \lesssim n^{11/4} k^{-15/4} + nk^{-1} + n^{13/2} p^{-15/2}.$$

On the other hand, Vinh’s estimate gives, whenever $k \gtrsim n/p$,

$$(3.19) \quad |\mathcal{L}_k| \lesssim pnk^{-2}.$$

We use the standard dyadic rectangle count from the proof of Lemma 2.4. Assign to each rectangle one of its edge-lines having maximal intersection with A . If $K \geq 1$, the rectangles assigned to lines containing at most K points contribute

$$(3.20) \quad \lesssim n^2 K.$$

For a dyadic class $k \geq K$, the contribution is at most

$$(3.21) \quad \begin{cases} k^3 |\mathcal{L}_k|, & K \leq k \leq n^{1/2}, \\ nk |\mathcal{L}_k|, & k \geq n^{1/2}. \end{cases}$$

Indeed, the first line follows by choosing an assigned edge, one endpoint, and the adjacent perpendicular edge. For the second line, the perpendicular lines through the points of a fixed edge are parallel and disjoint, so their intersections with A have total size at most n .

Take $K \sim p^{3/7}$, with the implicit constant large enough that $K \gtrsim n/p$ which we can do because $n \leq p^{10/7}$. First suppose that $n \geq p^{8/7}$. Applying (3.19) in (3.21), both richness ranges contribute at most

$$pn^{3/2} \leq n^2 p^{3/7}$$

per dyadic class. Together with (3.20), this proves the claim in this case.

It remains to consider $n \leq p^{8/7}$. For $K \leq k \leq n^{1/2}$, combine (3.18) and (3.19). The leading terms satisfy

$$\min\{n^{11/4}k^{-3/4}, pnk\} \leq (n^{11/4}k^{-3/4})^{4/7} (pnk)^{3/7} = n^2 p^{3/7}.$$

The other two terms from (3.18), after multiplication by k^3 , are bounded by

$$nk^2 + n^{13/2}p^{-15/2}k^3 \leq n^2 + n^8 p^{-15/2} \lesssim n^2 p^{3/7},$$

where the last inequality uses $n \leq p^{8/7}$. For $k \geq n^{1/2}$, (3.18) and the second line of (3.21) give

$$n^{15/4}k^{-11/4} + n^2 + n^{15/2}p^{-15/2}k \leq n^{19/8} + n^2 + n^{17/2}p^{-15/2} \lesssim n^2 p^{3/7},$$

again because $n \leq p^{8/7}$. Summing the $O(\log p)$ dyadic classes and absorbing the logarithm into p^ε proves (3.17). \square

COROLLARY 3.6. *Let $A, B \subset \mathbb{F}_p^2$ be k -regular sets, or both irregular sets, with $|A| \sim |B| \sim m$ and*

$$p \lesssim m \lesssim p^{10/7}.$$

Then, for every $\varepsilon > 0$,

$$(3.22) \quad \mathcal{B}(A, B) \lesssim_\varepsilon p^\varepsilon \left(m^{8/3} p^{2/3} + \frac{m^4}{p} + pm^2 + \min\{m^3 p^{1/3}, m^2 p^{10/7}\} \right).$$

PROOF. Since $m \gtrsim p$, Proposition 3.4 gives the right side of (3.22) with $m^3 p^{1/3}$ in place of the minimum. Lemma 3.5 and the second alternative in (2.15) give independently

$$\mathcal{B}(A, B) \leq p(\mathcal{R}(A)\mathcal{R}(B))^{1/2} \lesssim_\varepsilon p^\varepsilon m^2 p^{10/7}.$$

For nonnegative U, V, D one has $\min\{U + V, D\} \leq U + \min\{V, D\}$. Applying this with

$$U = m^{8/3} p^{2/3} + m^4 p^{-1} + pm^2, \quad V = m^3 p^{1/3}, \quad D = m^2 p^{10/7},$$

proves the result. \square

We insert Proposition 3.4 and Corollary 3.6 into the bilinear inequality. Write $N = |G|$. As in [9], after the standard dyadic reductions in the horizontal slices, and at the cost of a factor p^ε , it suffices to consider the following situation. The function $g \sim 1_G$, the set G has w nonempty horizontal slices, each slice has size

$$m \sim Nw^{-1},$$

and the planar supports G_z of all nonempty slices lie in one common dyadic regularity class. More precisely, either every G_z is k -regular for the same dyadic k , or every G_z is irregular, in which case we write $k \sim m^{1/2}$.

Indeed, after first pigeonholing the slice sizes, Lemma 2.5 decomposes each slice into $O(\log p)$ regular or irregular components. Write $g = \sum_\nu g_\nu$ for the resulting pieces, grouped by slice-size

class and regularity class. There are only $O(\log^2 p)$ such pieces, so by the triangle inequality it is enough to prove the restricted estimate for one g_ν and then sum, absorbing the logarithmic loss into p^ε . For that single piece, which we relabel as g , all nonempty slices have comparable size and lie in one common regularity class. Thus no cross-class bilinear terms remain, and Proposition 3.4 may be applied to every pair of nonempty slices.

We start from (2.16), with $q = p$, namely

$$(3.23) \quad \|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim N^{1/2} + N^{3/8} p^{-1/8} \left(\sum_z \mathcal{R}(G_z)^{1/2} + p^{-1/2} \sum_{z \neq z'} \mathcal{B}(G_z, G_{z'})^{1/2} + pN \right)^{1/4}.$$

We estimate the three terms inside the fourth root separately. For the linear slice term, (2.7) gives $\mathcal{R}(G_z) \lesssim m^{5/2}$ for each nonempty slice, and hence

$$(3.24) \quad \sum_z \mathcal{R}(G_z)^{1/2} \lesssim w m^{5/4} = w(Nw^{-1})^{5/4} = N^{5/4} w^{-1/4} \leq N^{5/4}.$$

The contribution of this term to (3.23) is therefore at most

$$N^{3/8} p^{-1/8} (N^{5/4})^{1/4} = p^{-1/8} N^{11/16}.$$

The last term inside the fourth root in (3.23) contributes

$$N^{3/8} p^{-1/8} (pN)^{1/4} = p^{1/8} N^{5/8}.$$

It remains to estimate the bilinear slice term. The same calculation will be used for each term in the bound for $\mathcal{B}(G_z, G_{z'})$. If one such term is $X(m, p)$, then after taking the square root and summing over the at most w^2 ordered pairs of nonempty slices, it gives

$$p^{-1/2} w^2 X(m, p)^{1/2}$$

inside the fourth root in (3.23). Thus its contribution to (3.23) is

$$(3.25) \quad N^{3/8} p^{-1/8} \left(p^{-1/2} w^2 X(m, p)^{1/2} \right)^{1/4} = p^{-1/4} N^{3/8} w^{1/2} X(m, p)^{1/8}.$$

We apply this calculation term by term to the bound in Proposition 3.4.

First take $X = m^{8/3} p^{2/3}$. Since $m = Nw^{-1}$ and $w \leq p$, (3.25) gives

$$(3.26) \quad \begin{aligned} p^{-1/4} N^{3/8} w^{1/2} (m^{8/3} p^{2/3})^{1/8} &= p^{-1/6} N^{3/8} w^{1/2} m^{1/3} \\ &= p^{-1/6} N^{17/24} w^{1/6} \\ &\leq N^{17/24}. \end{aligned}$$

Next take $X = m^4 p^{-1}$. Then

$$(3.27) \quad \begin{aligned} p^{-1/4} N^{3/8} w^{1/2} (m^4 p^{-1})^{1/8} &= p^{-3/8} N^{3/8} w^{1/2} m^{1/2} \\ &= p^{-3/8} N^{7/8}. \end{aligned}$$

Finally take $X = pm^2$. This gives

$$(3.28) \quad \begin{aligned} p^{-1/4} N^{3/8} w^{1/2} (pm^2)^{1/8} &= p^{-1/8} N^{3/8} w^{1/2} m^{1/4} \\ &= p^{-1/8} N^{5/8} w^{1/4} \\ &\leq p^{1/8} N^{5/8}. \end{aligned}$$

It remains to handle the summand $m^3 \min(m, p)^{1/3}$ in Proposition 3.4, together with the improvement supplied by Corollary 3.6. If $N \leq p^2$, then $m^3 \min(m, p)^{1/3} \leq m^{10/3}$. Using (3.25), its

contribution is at most

$$\begin{aligned}
(3.29) \quad & p^{-1/4} N^{3/8} w^{1/2} (m^{10/3})^{1/8} = p^{-1/4} N^{3/8} w^{1/2} m^{5/12} \\
& = p^{-1/4} N^{19/24} w^{1/12} \\
& \leq p^{-1/6} N^{19/24} \\
& \leq N^{17/24},
\end{aligned}$$

where the last two inequalities use $w \leq p$ and $N \leq p^2$. We may therefore assume from now on that $N \geq p^2$. Since $w \leq p$, this implies $m = Nw^{-1} \geq p$ on every nonempty slice.

For slice sizes in the range of Corollary 3.6, the remaining part of the bilinear input is

$$(3.30) \quad \min\{m^3 p^{1/3}, m^2 p^{10/7}\}.$$

The contribution obtained from the first term in (3.30) is

$$\begin{aligned}
(3.31) \quad & p^{-1/4} N^{3/8} w^{1/2} (m^3 p^{1/3})^{1/8} = p^{-5/24} N^{3/8} w^{1/2} m^{3/8} \\
& = p^{-5/24} N^{3/4} w^{1/8}.
\end{aligned}$$

The contribution obtained from the second term in (3.30) is

$$\begin{aligned}
(3.32) \quad & p^{-1/4} N^{3/8} w^{1/2} (m^2 p^{10/7})^{1/8} = p^{-1/14} N^{3/8} w^{1/2} m^{1/4} \\
& = p^{-1/14} N^{5/8} w^{1/4}.
\end{aligned}$$

The two inputs in (3.30) are equal when

$$(3.33) \quad m = p^{23/21}.$$

Equivalently, since $m = Nw^{-1}$, the critical number of slices is

$$(3.34) \quad w_0 = Np^{-23/21}.$$

For $w \geq w_0$ the first term in (3.30) is smaller, while for $w \leq w_0$ the second term is smaller.

We first dispose of the range $m \gtrsim p^{10/7}$, where Corollary 3.6 is not being used. This range corresponds to $w \lesssim Np^{-10/7}$. Using the contribution (3.31), we get

$$\begin{aligned}
(3.35) \quad & p^{-5/24} N^{3/4} w^{1/8} \lesssim p^{-5/24} N^{3/4} (Np^{-10/7})^{1/8} \\
& = p^{-65/168} N^{7/8} \\
& \leq p^{-3/8} N^{7/8}.
\end{aligned}$$

This is already covered by (3.27).

It remains to consider the range $m \lesssim p^{10/7}$. Suppose first that

$$(3.36) \quad p^2 \leq N \leq p^{44/21}.$$

Then $w_0 \leq p$. If $w \leq w_0$, we use the second contribution (3.32); by (3.34),

$$\begin{aligned}
(3.37) \quad & p^{-1/14} N^{5/8} w^{1/4} \leq p^{-1/14} N^{5/8} (Np^{-23/21})^{1/4} \\
& = p^{-29/84} N^{7/8} \\
& = p^{-1/12} N^{3/4} (p^{-11/42} N^{1/8}) \\
& \leq p^{-1/12} N^{3/4},
\end{aligned}$$

where the last inequality uses $N \leq p^{44/21}$. If instead $w \geq w_0$, we use the first contribution (3.31); since $w \leq p$,

$$(3.38) \quad p^{-5/24} N^{3/4} w^{1/8} \leq p^{-5/24} N^{3/4} p^{1/8} = p^{-1/12} N^{3/4}.$$

Thus the hybrid part contributes at most

$$(3.39) \quad p^{-1/12} N^{3/4} \quad (p^2 \leq N \leq p^{44/21}).$$

Now suppose that $N \geq p^{44/21}$. Then $w_0 \geq p$, so for all $w \leq p$ in the range $m \lesssim p^{10/7}$, the second term in (3.30) is the smaller one. Using (3.32) and $w \leq p$ gives

$$(3.40) \quad p^{-1/14} N^{5/8} w^{1/4} \leq p^{-1/14} N^{5/8} p^{1/4} = p^{5/28} N^{5/8}.$$

Combining (3.24), (3.26), (3.27), (3.28), (3.29), (3.35), (3.39), (3.40), and the diagonal term pN in (3.23), we obtain the following three estimates. If $N \leq p^2$, then

$$(3.41) \quad \|\widehat{g}\|_{L^2(P,d\sigma)} \lesssim_\varepsilon p^\varepsilon \left(N^{1/2} + p^{-1/8} N^{11/16} + N^{17/24} + p^{1/8} N^{5/8} + p^{-3/8} N^{7/8} \right).$$

If $p^2 \leq N \leq p^{44/21}$, then

$$(3.42) \quad \|\widehat{g}\|_{L^2(P,d\sigma)} \lesssim_\varepsilon p^\varepsilon \left(N^{1/2} + p^{-1/8} N^{11/16} + N^{17/24} + p^{1/8} N^{5/8} + p^{-3/8} N^{7/8} + p^{-1/12} N^{3/4} \right).$$

If $N \geq p^{44/21}$, then

$$(3.43) \quad \|\widehat{g}\|_{L^2(P,d\sigma)} \lesssim_\varepsilon p^\varepsilon \left(N^{1/2} + p^{-1/8} N^{11/16} + N^{17/24} + p^{1/8} N^{5/8} + p^{-3/8} N^{7/8} + p^{5/28} N^{5/8} \right).$$

We combine (3.41)–(3.43) with the small-set estimate

$$\|\widehat{g}\|_{L^2(P,d\sigma)} \lesssim N^{1/2} + Np^{-1/2}$$

and with the prime-field estimate (2.20) from [10]. The part of (2.20) that will be used below is the row

$$p^{3/16} N^{5/8}, \quad p^{75/34} \leq N \leq p^{5/2}.$$

The term $p^{-1/8} N^{11/16}$ is always bounded by $N^{17/24}$, and

$$(3.44) \quad p^{1/8} N^{5/8} \leq N^{17/24} \quad \text{whenever} \quad N \geq p^{3/2}.$$

Also

$$(3.45) \quad p^{-3/8} N^{7/8} \leq N^{17/24} \quad \iff \quad N \leq p^{9/4},$$

and

$$(3.46) \quad p^{5/28} N^{5/8} \leq N^{17/24} \quad \iff \quad N \geq p^{15/7}.$$

The middle hybrid term meets $N^{17/24}$ at $N = p^2$:

$$(3.47) \quad p^{-1/12} N^{3/4} = N^{17/24} \quad \iff \quad N = p^2.$$

The two hybrid terms meet at

$$(3.48) \quad p^{-1/12} N^{3/4} = p^{5/28} N^{5/8} \quad \iff \quad N = p^{44/21}.$$

Finally, since $75/34 < 9/4$, the row $p^{3/16} N^{5/8}$ in (2.20) is available for every N with $p^{9/4} \leq N \leq p^{5/2}$, and it agrees with $N^{17/24}$ when $N = p^{9/4}$. We therefore arrive at

$$(3.49) \quad \|\widehat{g}\|_{L^2(P,d\sigma)} \lesssim_\varepsilon p^\varepsilon \left(N^{1/2} + \begin{cases} Np^{-1/2}, & N \leq p^{12/7}, \\ N^{17/24}, & p^{12/7} \leq N \leq p^2, \\ p^{-1/12} N^{3/4}, & p^2 \leq N \leq p^{44/21}, \\ p^{5/28} N^{5/8}, & p^{44/21} \leq N \leq p^{15/7}, \\ N^{17/24}, & p^{15/7} \leq N \leq p^{9/4}, \\ p^{3/16} N^{5/8}, & p^{9/4} \leq N \leq p^{5/2}, \\ p^{1/2} N^{1/2}, & p^{5/2} \leq N \leq p^3. \end{cases} \right).$$

The estimate (3.49) was derived for a single dyadically reduced piece, whose nonempty slices have comparable size and lie in one common regularity class. Since the decomposition above has only $O(\log^2 p)$ pieces and we sum the pieces by the triangle inequality, the same restricted-type estimate, with the displayed p^ε loss, holds for the original function $g \sim 1_G$.

Set

$$\alpha = \frac{125}{176}.$$

Every entry of (3.49) is bounded by N^α in its stated range. For the first entry,

$$(3.50) \quad Np^{-1/2} \leq N^\alpha \iff N \leq p^{88/51},$$

and $12/7 < 88/51$. The pure power $N^{17/24}$ is admissible because $17/24 < 125/176$. For the two new entries,

$$(3.51) \quad p^{-1/12}N^{3/4} \leq N^\alpha \iff N \leq p^{44/21},$$

while

$$(3.52) \quad p^{5/28}N^{5/8} \leq N^\alpha \iff N \geq p^{44/21}.$$

For the term imported from (2.20),

$$(3.53) \quad p^{3/16}N^{5/8} \leq N^\alpha \iff N \geq p^{11/5},$$

and $9/4 > 11/5$. Lastly, $p^{1/2}N^{1/2} \leq N^{7/10} < N^\alpha$ for $N \geq p^{5/2}$. Hence

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim_\varepsilon p^\varepsilon N^{125/176}.$$

By Lemma 2.2, (2.1) holds for every $s < 176/125$. Dualizing gives

$$R^*(2 \rightarrow r) \lesssim_r 1 \quad \text{for every } r > \frac{176}{51},$$

which proves Theorem 1.1.

4. The endpoint estimate in six dimensions

Let $P = P_6 = \{(\xi, \xi \cdot \xi) : \xi \in F^5\} \subset F^6$. We prove $R^*(2 \rightarrow \frac{8}{3}) \lesssim 1$. The only nontrivial input is the additive-energy estimate of [7], namely

$$(4.1) \quad \Lambda(E) \lesssim q^{-1}|E|^3 + q^2|E|^2, \quad E \subset P_6,$$

which is sharp in general because P_6 contains affine two-planes. We write $\mathcal{Q}(f, h) = \langle \widehat{f}, \widehat{h} \rangle_{L^2(P, d\sigma)} = \langle f, h * (d\sigma)^\vee \rangle_{F^6}$ for the associated bilinear form. The argument below is bilinear, of the type used in [13] to reach endpoint estimates without a logarithmic loss.

PROPOSITION 4.1. *Let $A, B \subset F^6$ and let $|f| \leq 1_A$, $|h| \leq 1_B$. Then*

$$(4.2) \quad |\mathcal{Q}(f, h)| \lesssim |A \cap B| + q^{-1}|A|^{3/4}|B|^{3/4} + |A|^{3/4}|B|^{1/2},$$

and consequently

$$(4.3) \quad |\mathcal{Q}(f, h)| \lesssim |A \cap B| + q^{-1}|A|^{3/4}|B|^{3/4} + \min\{|A|^{3/4}|B|^{1/2}, |A|^{1/2}|B|^{3/4}\}.$$

We also have the (cruder) estimate

$$(4.4) \quad |\mathcal{Q}(f, h)| \lesssim q|A|^{1/2}|B|^{1/2}.$$

PROOF. Since $(d\sigma)^\vee = \delta_0 + K$, the contribution of δ_0 to $\mathcal{Q}(f, h) = \langle f, h * (d\sigma)^\vee \rangle$ is $\langle f, h \rangle$, of modulus at most $|A \cap B|$. For the contribution of K , Hölder's inequality with exponents $\frac{4}{3}$ and 4 gives

$$(4.5) \quad |\langle f, h * K \rangle| \leq \|f\|_{4/3} \|h * K\|_4 \leq |A|^{3/4} \|h * K\|_4.$$

To estimate $\|h * K\|_4$ we use the slice estimate of Lemma 2.1 with $d = 6$. For $z \in F$ let $h_z(\eta) = h(\eta, z)$ for $\eta \in F^5$, supported on $B_z = \{\eta \in F^5 : (\eta, z) \in B\}$, and let \tilde{h}_z be its lift to the paraboloid, defined by $\tilde{h}_z(\eta, \eta \cdot \eta) = h_z(\eta)$ on its support

$$\tilde{B}_z = \{(\eta, \eta \cdot \eta) : \eta \in B_z\} \subset P,$$

so that $|\tilde{B}_z| = |B_z|$ and $\sum_z |B_z| = |B|$. Lemma 2.1 gives

$$(4.6) \quad \|h * K\|_4 \lesssim q^{5/2} \sum_z \|(\tilde{h}_z d\sigma)^\vee\|_4.$$

For any function $\phi : P \rightarrow \mathbb{C}$ with $|\phi| \leq 1_E$ for some set $E \subset P$, expanding the fourth power and using Plancherel on F^6 identifies $\|(\phi d\sigma)^\vee\|_4^4$ with a normalized weighted count of additive quadruples in E , the quadruple $(\xi_1, \xi_2, \xi_3, \xi_4)$ with $\xi_1 + \xi_2 = \xi_3 + \xi_4$ carrying the weight $\phi(\xi_1)\phi(\xi_2)\overline{\phi(\xi_3)}\overline{\phi(\xi_4)}$. Since $|\phi| \leq 1_E$, each weight has modulus at most 1, so this is at most the unweighted additive energy $\Lambda(E)$, whence

$$(4.7) \quad \|(\phi d\sigma)^\vee\|_4 \leq q^{-7/2} \Lambda(E)^{1/4}.$$

Applying the energy estimate (4.1) to $E = \tilde{B}_z$ and $\phi = \tilde{h}_z$ gives $\Lambda(\tilde{B}_z)^{1/4} \lesssim q^{-1/4} |B_z|^{3/4} + q^{1/2} |B_z|^{1/2}$, so combining (4.6) and (4.7),

$$(4.8) \quad \|h * K\|_4 \lesssim q^{-1} \sum_z \Lambda(\tilde{B}_z)^{1/4} \lesssim q^{-5/4} \sum_z |B_z|^{3/4} + q^{-1/2} \sum_z |B_z|^{1/2}.$$

There are at most q nonempty slices, so Hölder's inequality gives $\sum_z |B_z|^{3/4} \leq q^{1/4} |B|^{3/4}$ and $\sum_z |B_z|^{1/2} \leq q^{1/2} |B|^{1/2}$, and the right side of (4.8) reduces to

$$(4.9) \quad \|h * K\|_4 \lesssim q^{-1} |B|^{3/4} + |B|^{1/2}.$$

Substituting (4.9) into (4.5) gives (4.2), and interchanging the roles of f and h and keeping the smaller of the two bounds gives (4.3). Finally, by Plancherel on F^6 ,

$$\|\widehat{f}\|_{L^2(P, d\sigma)}^2 = q^{-5} \sum_{\xi \in P} |\widehat{f}(\xi)|^2 \leq q^{-5} \sum_{\xi \in F^6} |\widehat{f}(\xi)|^2 = q \|f\|_2^2,$$

so the Cauchy-Schwarz inequality $|\mathcal{Q}(f, h)| \leq \|\widehat{f}\|_{L^2(P, d\sigma)} \|\widehat{h}\|_{L^2(P, d\sigma)}$ gives (4.4). \square

We now prove the endpoint. By duality, $R^*(2 \rightarrow \frac{8}{3}) \lesssim 1$ is equivalent to $\|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim \|g\|_{L^{8/5}(F^6)}$, the exponent $\frac{8}{5}$ being conjugate to $\frac{8}{3}$. Normalize $\sum_x |g(x)|^{8/5} = 1$, so that $|g| \leq 1$, and decompose g into dyadic level sets $G_i = \{x : 2^{-i-1} < |g(x)| \leq 2^{-i}\}$ for $i \geq 0$, writing $g_i = g 1_{G_i} = 2^{-i} h_i$ with $|h_i| \leq 1_{G_i}$. Set

$$a_i = 2^{-8i/5} |G_i|.$$

Then $\sum_i a_i \lesssim 1$, and in particular $a_i \lesssim 1$ for every i . We will use the identity

$$|G_i| = 2^{8i/5} a_i$$

each time a cardinality $|G_i|$ is rewritten below. Expanding the square gives

$$(4.10) \quad \|\widehat{g}\|_{L^2(P, d\sigma)}^2 = \sum_{i, j \geq 0} 2^{-i-j} \mathcal{Q}(h_i, h_j) \leq \sum_{i, j \geq 0} 2^{-i-j} |\mathcal{Q}(h_i, h_j)|.$$

Inserting the symmetric bound (4.3), with $A = G_i$ and $B = G_j$, into (4.10) gives the three terms

$$(4.11) \quad \sum_{i,j \geq 0} 2^{-i-j} |G_i \cap G_j| + q^{-1} \sum_{i,j \geq 0} 2^{-i-j} |G_i|^{3/4} |G_j|^{3/4} \\ + \sum_{i,j \geq 0} 2^{-i-j} \min \{ |G_i|^{3/4} |G_j|^{1/2}, |G_i|^{1/2} |G_j|^{3/4} \}.$$

We will bound the three displayed terms separately. The first and third terms can be directly summed. The middle term is more subtle at the endpoint, and we will also use the Parseval estimate (4.4) for each pair of dyadic levels.

The first term contributes only when $i = j$, since the level sets G_i are disjoint. Thus

$$(4.12) \quad \sum_{i,j \geq 0} 2^{-i-j} |G_i \cap G_j| = \sum_i 2^{-2i} |G_i| = \sum_i 2^{-2i} (2^{8i/5} a_i) = \sum_i 2^{-2i/5} a_i \leq \sum_i a_i \lesssim 1.$$

Next consider the last term in (4.11). Suppose first that $i \leq j$, and use the first term of the minimum. This gives us:

$$2^{-i-j} |G_i|^{3/4} |G_j|^{1/2} = 2^{-i-j} (2^{8i/5} a_i)^{3/4} (2^{8j/5} a_j)^{1/2} \\ = 2^{-(j-i)/5} a_i^{3/4} a_j^{1/2} \\ \lesssim 2^{-(j-i)/5} a_i^{1/2} a_j^{1/2},$$

where the last inequality uses $a_i \lesssim 1$. The case $j \leq i$ is identical, using the second term of the minimum. Therefore the total contribution of the last term in (4.11) is at most

$$\sum_{i,j \geq 0} 2^{-|i-j|/5} a_i^{1/2} a_j^{1/2}.$$

Applying $a_i^{1/2} a_j^{1/2} \leq \frac{1}{2}(a_i + a_j)$ and using the symmetry in i and j gives

$$(4.13) \quad \sum_{i,j \geq 0} 2^{-|i-j|/5} a_i^{1/2} a_j^{1/2} \leq \sum_{i,j \geq 0} 2^{-|i-j|/5} a_i = \sum_{i \geq 0} a_i \sum_{j \geq 0} 2^{-|i-j|/5} \lesssim \sum_i a_i \lesssim 1.$$

It remains to estimate the middle term in (4.11). For a fixed pair (i, j) we have:

$$q^{-1} 2^{-i-j} |G_i|^{3/4} |G_j|^{3/4} = q^{-1} 2^{-i-j} (2^{8i/5} a_i)^{3/4} (2^{8j/5} a_j)^{3/4} = q^{-1} 2^{(i+j)/5} a_i^{3/4} a_j^{3/4} \\ \lesssim q^{-1} 2^{(i+j)/5} a_i^{1/2} a_j^{1/2}.$$

This estimate grows with $i + j$, so it cannot be summed directly. For the same pair (i, j) , however, the crude Parseval estimate (4.4) gives

$$2^{-i-j} |\mathcal{Q}(h_i, h_j)| \lesssim q 2^{-i-j} |G_i|^{1/2} |G_j|^{1/2}.$$

Again substituting the cardinalities and then simplifying, the right-hand side is

$$q 2^{-i-j} |G_i|^{1/2} |G_j|^{1/2} = q 2^{-i-j} (2^{8i/5} a_i)^{1/2} (2^{8j/5} a_j)^{1/2} = q 2^{-(i+j)/5} a_i^{1/2} a_j^{1/2}.$$

More precisely, for each pair (i, j) the bound coming from (4.3) and the Parseval bound (4.4) imply that the contribution of this pair is bounded by the diagonal and asymmetric contributions from (4.11), plus the minimum of the middle estimate above and the Parseval estimate. Thus the total contribution of the middle term is bounded by

$$\sum_{i,j \geq 0} a_i^{1/2} a_j^{1/2} \min \{ q^{-1} 2^{(i+j)/5}, q 2^{-(i+j)/5} \}.$$

Using again $a_i^{1/2}a_j^{1/2} \leq \frac{1}{2}(a_i + a_j)$ and the symmetry in i and j , this is at most

$$\sum_{i,j \geq 0} a_i \min \{q^{-1}2^{(i+j)/5}, q2^{-(i+j)/5}\}.$$

For each fixed i , the inner sum over j is bounded by the full one-dimensional sum

$$\sum_{n \geq 0} \min \{q^{-1}2^{n/5}, q2^{-n/5}\}.$$

Let n_0 be an integer satisfying $2^{n_0/5} \leq q < 2^{(n_0+1)/5}$. Then

$$\begin{aligned} \sum_{n \geq 0} \min \{q^{-1}2^{n/5}, q2^{-n/5}\} &\leq q^{-1} \sum_{0 \leq n \leq n_0} 2^{n/5} + q \sum_{n > n_0} 2^{-n/5} \\ &\lesssim q^{-1}2^{n_0/5} + q2^{-(n_0+1)/5} \lesssim 1. \end{aligned}$$

Therefore

$$(4.14) \quad \sum_{i,j \geq 0} a_i \min \{q^{-1}2^{(i+j)/5}, q2^{-(i+j)/5}\} \lesssim \sum_i a_i \lesssim 1.$$

Combining (4.12), (4.13), and (4.14) gives $\|\widehat{g}\|_{L^2(P,d\sigma)}^2 \lesssim 1$. Since we normalized $\|g\|_{L^{8/5}(F^6)} = 1$, this proves $\|\widehat{g}\|_{L^2(P,d\sigma)} \lesssim \|g\|_{L^{8/5}(F^6)}$. By duality, this is equivalent to $R^*(2 \rightarrow \frac{8}{3}) \lesssim 1$, which proves Theorem 1.2.

5. A transference observation for lattice $\Lambda(p)$ estimates

We close with an observation that connects the three-dimensional endpoint conjecture with the analogous problem in lattices. This connection is not particularly deep, but seems to have gone unnoticed in the prior literature.

Consider first the integer paraboloid $\Gamma = \{(n_1, n_2, n_1^2 + n_2^2) : n_1, n_2 \in \mathbb{Z}\} \subset \mathbb{Z}^3$, and suppose the endpoint finite field estimate

$$(5.1) \quad \|(fd\sigma)^\vee\|_{L^3(\mathbb{F}_p^3)} \lesssim \|f\|_{L^2(P_3,d\sigma)}$$

holds uniformly for the three-dimensional paraboloid over all sufficiently large primes p in a congruence class for which -1 is not a square. Let $S \subset \mathbb{Z}^2$ be finite, let $(a_n)_{n \in S}$ be complex coefficients, and choose primes $p \rightarrow \infty$ so large that reduction modulo p is injective on $\{(n_1, n_2, n_1^2 + n_2^2) : n \in S\}$. Writing (5.1) in unnormalized form, and noting that the normalization cancels exactly because $|P_3| \sim p^2$ while the ambient space has size p^3 , gives, with $e_p(t) = e^{2\pi it/p}$ the standard additive character of \mathbb{F}_p ,

$$\left(p^{-3} \sum_{x \in \mathbb{F}_p^3} \left| \sum_{n \in S} a_n e_p(x_1 n_1 + x_2 n_2 + x_3 (n_1^2 + n_2^2)) \right|^3 \right)^{1/3} \lesssim \left(\sum_{n \in S} |a_n|^2 \right)^{1/2}.$$

Letting $p \rightarrow \infty$, the left-hand side converges to the Riemann integral over $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ of the corresponding trigonometric sum, in which $e(t) = e^{2\pi it}$, so the finite field endpoint would imply

$$(5.2) \quad \left\| \sum_{n \in S} a_n e(n_1 t_1 + n_2 t_2 + (n_1^2 + n_2^2) t_3) \right\|_{L^3(\mathbb{T}^3)} \lesssim \left(\sum_{n \in S} |a_n|^2 \right)^{1/2},$$

with a constant independent of the finite set S . Since the constant is independent of the finite set S , the extension operator initially defined on finitely supported coefficient sequences on Γ is bounded from $\ell^2(\Gamma)$ to $L^3(\mathbb{T}^3)$. By the bounded linear extension theorem it extends to all of $\ell^2(\Gamma)$, and Plancherel identifies the extension with the corresponding Fourier series. Thus the full integer paraboloid Γ is a $\Lambda(3)$ set.

The same argument applies to spheres. Let $S_R = \{n \in \mathbb{Z}^3 : n_1^2 + n_2^2 + n_3^2 = R\}$ be a lattice sphere, and suppose the finite field endpoint holds uniformly for non-degenerate spheres in \mathbb{F}_p^3 . If $R = 0$ there is nothing to prove, so assume $R \neq 0$. For a fixed finite subset of S_R , choosing primes p so large that reduction is injective and R remains a non-degenerate radius modulo p , and passing to the limit, gives

$$\left\| \sum_{n \in S_R} a_n e(n \cdot t) \right\|_{L^3(\mathbb{T}^3)} \lesssim \left(\sum_{n \in S_R} |a_n|^2 \right)^{1/2},$$

uniformly in the radius and in the finite support. Thus each fixed lattice sphere S_R would be a $\Lambda(3)$ set, with a constant independent of R .

This phenomenon is special to ambient dimension three. If $V \subset \mathbb{F}_q^d$ has size $\sim q^{d-1}$ and $R^*(2 \rightarrow r) \lesssim 1$, then the corresponding normalized exponential-sum estimate, with e_q the canonical additive character of \mathbb{F}_q , reads

$$(5.3) \quad \left(q^{-d} \sum_{x \in \mathbb{F}_q^d} \left| \sum_{\xi \in V} a_\xi e_q(x \cdot \xi) \right|^r \right)^{1/r} \lesssim q^{(d-1)/2-d/r} \left(\sum_{\xi \in V} |a_\xi|^2 \right)^{1/2},$$

with no power of q exactly when $r = \frac{2d}{d-1}$, which for $d = 3$ is the conjectured finite field endpoint $r = 3$. In higher dimensions the finite field L^2 endpoint does not coincide with $\frac{2d}{d-1}$, so a power of q remains; the passage to a loss-free $\Lambda(3)$ estimate is thus particular to $d = 3$.

The exponent 3 obtained here falls short of the full conjecture. The discrete restriction conjecture of Bourgain [4] predicts that the integer paraboloid is a $\Lambda(r)$ set for every $r < 4$, and that each lattice sphere is a $\Lambda(r)$ set for every $r < 6$, the wider range for the sphere reflecting the greater arithmetic sparsity of the discrete sphere [1, 3]. For the paraboloid the exponent $r = 4$ is critical, so the estimate obtained here is subcritical. The critical paraboloid exponent is known up to a loss of N^ε in the size N of the support, by the ℓ^2 decoupling theorem of Bourgain and Demeter [2], which gives $r = 4$; the corresponding endpoint $r = 6$ for the sphere remains open even with such a loss. Removing this N^ε loss, so as to reach any subcritical exponent without it, appears to require substantial new ideas, and no loss-free $\Lambda(r)$ estimate is known for any surface for any $r > 2$ unless it is implied by $r = 4$; see [3] for a related discussion.

Statement on AI usage

The proof presented here is due to the author. As noted in the footnote in the introduction, the basic approach used in the proof was already known to the author when the earlier paper was written and was mentioned there, but it was not pursued because of confusion over the dominant term. The author used ChatGPT to help organize the argument and identify the cutoffs used to optimize the estimate. The author also used ChatGPT, Gemini, and Claude for assistance with formatting, copyediting, and proofreading.

References

- [1] J. Bourgain, *Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces*, Israel J. Math. **193** (2013), no. 1, 441–458.
- [2] J. Bourgain and C. Demeter, *The proof of the ℓ^2 decoupling conjecture*, Ann. of Math. (2) **182** (2015), no. 1, 351–389.
- [3] J. Bourgain, *Λ_p -sets in analysis: results, problems and related aspects*, in Handbook of the Geometry of Banach Spaces, Vol. I, 195–232, North-Holland, Amsterdam, 2001.
- [4] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, Part I: Schrödinger equations*, Geom. Funct. Anal. **3** (1993), no. 2, 107–156.
- [5] J. Bourgain, *Bounded orthogonal systems and the $\Lambda(p)$ -set problem*, Acta Math. **162** (1989), no. 3–4, 227–245.
- [6] A. Iosevich and D. Koh, *Extension theorems for paraboloids in the finite field setting*, Math. Z. **266** (2010), 471–487.

- [7] A. Iosevich, D. Koh and M. Lewko, *Finite field restriction estimates for the paraboloid in high even dimensions*, J. Funct. Anal. **278** (2020), no. 11, 108450.
- [8] J. Kollár, *Szemerédi–Trotter-type theorems in dimension 3*, Adv. Math. **271** (2015), 30–61.
- [9] M. Lewko, *A bilinear approach to the finite field restriction problem*, arXiv:2408.03514.
- [10] M. Lewko, *Counting rectangles and an improved restriction estimate for the paraboloid in \mathbb{F}_p^3* , Proc. Amer. Math. Soc. **148** (2020), no. 4, 1535–1543.
- [11] M. Lewko, *Finite field restriction estimates based on Kakeya maximal operator estimates*, J. Eur. Math. Soc. **21** (2019), no. 12, 3649–3707.
- [12] M. Lewko, *New restriction estimates for the 3-d paraboloid over finite fields*, Adv. Math. **270** (2015), 457–479.
- [13] A. Lewko and M. Lewko, *Endpoint restriction estimates for the paraboloid over finite fields*, Proc. Amer. Math. Soc. **140** (2012), no. 6, 2013–2028.
- [14] B. Lund, T. Pham and L. A. Vinh, *Orthogonal projections and incidence bounds in planes over prime order fields*, arXiv:2311.05148 [math.CO].
- [15] G. Mockenhaupt and T. Tao, *Restriction and Kakeya phenomena for finite fields*, Duke Math. J. **121** (2004), no. 1, 35–74.
- [16] M. Rudnev, *On the number of incidences between points and planes in three dimensions*, Combinatorica **38** (2018), 219–254.
- [17] M. Rudnev and I. D. Shkredov, *On the restriction problem for discrete paraboloid in lower dimension*, Adv. Math. **339** (2018), 657–671.
- [18] M. Rudnev, *Point-plane incidences and some applications in positive characteristic*, arXiv:1806.03534.
- [19] S. Stevens and F. de Zeeuw, *An improved point-line incidence bound over arbitrary fields*, Bull. Lond. Math. Soc. **49** (2017), no. 5, 842–858.
- [20] L. Vinh, *The Szemerédi–Trotter type theorem and the sum-product estimate in finite fields*, European J. Combin. **32** (2011), no. 8, 1177–1181.

LEBANON, NEW HAMPSHIRE USA
 Email address: mlewko@gmail.com